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THE ERROR IN LEAST SQUARE SHAPING FILTER

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ABSTRACT. In least-squares inverse filtering, Claerbout and Robinson (1963) proved that, under certain conditions, the error will go to zero as the length of the filter tends to infinity.

In this paper, this result is extended to the case of the shaping filter when the desired output permits a delay.

INTRODUCTION

The problem of finding a filter that approximates in the least square sense a source wavelet w to a desired output d is known in signal processing. This filter is called shaping filter.

In inverse filtering, we deal with the case when $d = e_k = (\overbrace{0,\ldots,0}^{k},1,0,\ldots,0)$ (spiking filter). Claerbout and Robinson [1] have proved that in this case the spiking filter error will go to zero when the length of the filter tends to infinity. In this paper we show that this result can be extended to the shaping filter.

NOTATION

If $a \in R^{n+1}$, $a = (a_0,a_1,\ldots,a_n)$, we define $A_\ell \in R^{(\ell+1)\times(n+\ell+1)}$ as

$$A_\ell = \begin{pmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_n \end{pmatrix} \qquad (\ell = 0,1,2,\ldots) \tag{1}$$

In particular, A_0 is a, and if $a \in R$, then $A_\ell = a.I$ (I identity matrix).

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CONVOLUTION

Let $a = (a_0, \dots, a_n)$, $b = (b_0, \dots, b_m)$.

If we define $c = a * b = (c_0, \dots, c_{n+m})$ with

$$c_t = \sum_k a_k \, b_{t-k} \tag{2}$$

then $c = a.B_n$ and also $c = b.A_m$.

CORRELATION

If $a = (a_0, \dots, a_n)$, $d = (d_0, \dots, d_{n+\ell})$ $(\ell = 0,1,2,\dots)$ and $c_s = \sum_j d_{j+s} \, a_j$ $(s = 0,1,\dots,\ell)$ then, if ${}^tA$ denotes the transpose of a matrix A,

$$c = d \; {}^tA_\ell \tag{3}$$

If $a = (a_0, \dots, a_n)$ we define the matrix of the first ℓ autocorrelations of a as $R = (r_{i-j})$, $i,j = 0,\dots,\ell$, where

$$r_{i-j} = \sum_k a_{k+i} \, a_{k+j} = \sum_k a_k \, a_{k+(i-j)} .$$

Hence

$$R = A_\ell \; {}^tA_\ell \tag{4}$$

R is symmetrical and, if $a \neq 0$, non singular.

It also holds that, if $a = (a_0, \dots, a_n)$, $b = (b_0, \dots, b_m)$, $= (c_0, \dots, c_h)$ then

$$(a * b)_\ell = (a.B_n)_\ell = A_\ell \, B_{n+\ell} \tag{5}$$

and therefore

$$a*(b*c) = a.(b*c)_n = a.(b.C_m)_n = a.B_n \, C_{m+n} . \tag{6}$$

Furthermore, if $x = (x_0, \dots, x_n)$, $\|x\|_2 = (\sum x_i^2)^{1/2}$ and $\|x\|_1 = \sum |x_i|$ where $|x_i|$ is the absolute value of x_i.

SHAPING AND SPIKING FILTER

Let $w = (w_0, \dots, w_n)$ be a source wavelet and $d = (d_0, \dots, d_{n+\ell})$ the desired output; the Shaping filter of length $\ell+1$ is defined by the filter $f^0 = (f_0, f_1, \dots, f_\ell)$ that minimizes

$$\|w*f-d\|_2^2 \quad \text{for} \quad f \in R^{\ell+1} . \tag{7}$$

It is known that f^0 satisfies

$$f^0 \, W_\ell \, {}^tW_\ell = d \, {}^tW_\ell . \tag{8}$$

In particular, if $d = e_k = (\overbrace{0,\dots,0}^{k},1,0,\dots,0)$, $e_k \in R^{n+\ell+1}$, $k = 0,\dots,n+\ell$, the filter $a^k = (a_{k0},\dots,a_{k\ell})$ that minimizes

$$\|w*a-e_k\|_2^2 \quad \text{for} \quad a \in R^{\ell+1} \tag{9}$$

is called the k-delay Spiking filter.

Also a^k satisfies

$$a^k \, W_\ell \, {}^tW_\ell = e_k \, {}^tW_\ell \tag{10}$$

where $e_k \, {}^tW_\ell$ is the row (k+1) of the matrix ${}^tW_\ell$, that is $e_k \, {}^tW_\ell = (w_k, w_{k-1}, \dots, w_{k-\ell})$ with $w_i = 0$ if $i \notin [0,n]$.

If A is now the $(n+\ell+1)\times(\ell+1)$ matrix whose rows are the vectors $a^0,\dots,a^{n+\ell}$, it is known that the Shaping filter f^0 for the input w and the output d is

$$f^0 = dA \tag{11}$$

that is

$$f^0 = \sum_{k=0}^{n+\ell} d_k . a^k \tag{12}$$

(see [2, p.199]).

Then it results that the Shaping filter is a linear combination of the Spiking filters whose coefficients are the coordinates of the output d.

THE ERROR FOR THE SPIKING FILTER

Let w, a^k and e_k $(k = 0,\dots,n+\ell)$ be as in (9) and let us call

$$J_k = \|w*a^k - e_k\|_2^2$$

the error of the Spiking filter of delay k.

Claerbout and Robinson [1] have proved that

$$J_0+J_1+ \dots +J_{n+\ell} = n \quad , \quad 0 \leqslant J_k \leqslant 1 \tag{13}$$

that is, the sum of the errors of the Spiking filters is equal to the length of the wavelet minus one, and therefore independent of the filter length.

Hence, there exists k_0 so that

$$J_{k_0} \leq \frac{n}{n+\ell+1} \longrightarrow 0 \ . \qquad (14)$$

If V_ℓ is the minimum error of all the Spiking filters of length $\ell+1$

$$0 \leq V_\ell \leq \frac{n}{n+\ell+1} \longrightarrow 0 \qquad (\ell \to +\infty)$$

and then

$$V_\ell \longrightarrow 0 \qquad (\ell \to +\infty) \qquad (15)$$

A value of k that produces the minimum J_k is called the optimum delay or optimum spike position, and the corresponding spiking filter a^k is called the optimum spiking filter for the given wavelet w and filter length $\ell+1$.

In the case where w is minimum-phase, it is known [1] that

$$J_0 \longrightarrow 0 \qquad (\ell \to +\infty) \qquad (16)$$

THE ERROR FOR THE SHAPING FILTER

Let a^k be the Spiking filter of length $\ell+1$ and $c^k = a^k * w$ be the output of this filter, then

$$c^k = a^k . W_\ell \quad \text{and} \quad C = A\ W_\ell \qquad (17)$$

where C is the $(n+\ell+1)x(n+\ell+1)$ matrix with rows c^k and A is the matrix of the Spiking filters.

As a^k satisfies (10), that is $a^k\ W_\ell\ {}^tW_\ell = e_k\ {}^tW_\ell$, then

$A\ W_\ell\ {}^tW_\ell = {}^tW_\ell$, and multiplying on the right by ${}^tA$,

$A\ W_\ell\ {}^tW_\ell\ {}^tA = {}^tW_\ell\ {}^tA$, that is

$$C\ {}^tC = {}^tC \ . \qquad (18)$$

If now f^0 is the Shaping filter for the output d from (11) $f^0 = dA$ and from (8) $f^0\ W_\ell\ {}^tW_\ell = d\ {}^tW_\ell$.

The error $J = \|w*f^0-d\|_2^2$ is $(w*f^0 - d)\ {}^t(w*f^0 - d) =$

$= (dA*w - d)\ {}^t(dA*w - d) = (dA\ W_\ell - d)\ {}^t(dA\ W_\ell - d)$ and using

(18) and $C = AW_\ell$

$$J = d(I - C)\ {}^t d \quad , \ ([2, p.199]) \qquad (19)$$

which gives a simplified expression of the error for the Shaping filter.

AN ESTIMATE OF THE ERROR DEPENDING ON THE FILTER LENGTH

In this section we prove that the error of the shaping filter tends to zero when the length of the filter tends to infinity. This new result generalizes the known one for spiking filter.

We consider $w = (w_0,\dots,w_n)$, $d = (d_0,\dots,d_{n+\ell})$, $e_k \in R^{m+1}$

$e_k = (\overbrace{0,\dots,0}^{k},1,0,\dots,0)$, $k = 0,\dots,m$, and let f^k be the Shaping filter corresponding to the input w and the output $d*e_k$ of length $n+\ell+m+1$

$$f^k = (f_{k0}, f_{k1},\dots,f_{k\ell+m})$$

and $\varepsilon_k = \|w*f^k - d*e_k\|_2 \qquad k = 0,1,\dots,m.$

For the error ε, where

$$\varepsilon = \sum_{k=0}^{m} \varepsilon_k \qquad (20)$$

we have the following estimate

$$\varepsilon = \sum_{k=0}^{m} \|w*f^k - d*e_k\|_2 = \sum_{k=0}^{m} \|w*(\sum_{j=0}^{n+\ell} d_j\ a^{j+k}) - \sum_{j=0}^{n+\ell} d_j\ \delta_{k+j}\|_2 \ ,$$

with

$$\delta_h : [0,n+\ell+m] \to [0,1] \quad , \quad \delta_h(i) = \begin{cases} 1 & \text{if} \quad i = h \\ 0 & \text{if} \quad i \neq h \end{cases} .$$

Then

$$\varepsilon = \sum_{k=0}^{m} \| \sum_{j=0}^{n+\ell} d_j (w*a^{j+k}) - \sum_{j=0}^{n+\ell} d_j\ \delta_{j+k}\|_2 =$$

$$= \sum_{k=0}^{m} \| \sum_{j=0}^{n+\ell} d_j (w*a^{j+k} - \delta_{j+k})\|_2 \leqslant$$

$$\leqslant \sum_{k=0}^{m} (\sum_{j=0}^{n+\ell} |d_j|\ \|w*a^{j+k} - \delta_{j+k}\|_2) =$$

$$= \sum_{j=0}^{n+\ell} |d_j| (\sum_{k=0}^{m} \|w*a^{j+k} - \delta_{j+k}\|_2) \leqslant$$

(using Cauchy-Schwartz for the sum in the parenthesis)

$$\leq \sum_{j=0}^{n+\ell} |d_j| (m+1)^{1/2} \left(\sum_{k=0}^{m} \| w*a^{j+k} - \delta_{j+k} \|_2^2 \right)^{1/2} .$$

Now from (13) it follows that

$$\sum_{k=0}^{m} \| w*a^{j+k} - \delta_{j+k} \|_2^2 \leq \sum_{h=0}^{n+\ell+m} \| w*a^h - \delta_h \|_2^2 = n.$$

Then $$\varepsilon \leq \sum_{j=0}^{n+\ell} |d_j| (m+1)^{1/2} n^{1/2} = \| d \|_1 (m+1)^{1/2} n^{1/2}.$$

That is

$$\varepsilon = \sum_{k=0}^{m} \varepsilon_k \leq \| d \|_1 (m+1)^{1/2} n^{1/2} \qquad (21)$$

Then there exists $k_0 \in [0,m]$ such that

$$\varepsilon_{k_0} = \| w*f^{k_0} - d*e_{k_0} \|_2 \leq \| d_1 \| \frac{(m+1)^{1/2} n^{1/2}}{m+1}$$

hence $$\varepsilon_{k_0} \leq \| d \|_1 \cdot \frac{\sqrt{n}}{\sqrt{m+1}} \qquad (22)$$

If $\varepsilon_{min}(m)$ is the minimum error for the Shaping filter of length $\ell+m+1$ for the input $w = (w_0,\ldots,w_n)$ and the output $d*e_k = (\overbrace{0,\ldots,0}^{k},d_0,\ldots,d_{n+\ell},0,\ldots,0)$ we have that

$$\varepsilon_{min}(m) \to 0 \qquad (m \to +\infty). \qquad (23)$$

This result can also be obtained in the following way:
Let a^k be the optimum spiking filter of length $\ell+1$ for the input w. Then $\| w*a^k - e_k \|_2^2 \leq \frac{n}{n+\ell+1}$. Let now $d = (d_0,\ldots,d_m)$ be the desired output. Then $\| w*(a^k*d) - d*e_k \|_2 = \| d*(w*a^k - e_k) \|_2 \leq$

$$\leq \| d \|_1 \| w*a^k - e_k \|_2 \leq \| d \|_1 \frac{\sqrt{n}}{\sqrt{n+\ell+1}}$$

and if $f = a^k*d$ and f^k is the Shaping filter for the output $d*e_k$, it follows that

$$\varepsilon_k = \| w*f^k - d*e_k \|_2 \leq \| w*f - d*e_k \|_2 \leq \| d \|_1 \frac{\sqrt{n}}{\sqrt{n+\ell+1}} .$$

Thus $\varepsilon_{min}(\ell) \to 0 \quad (\ell \to +\infty)$.

In the case where w is minimum-phase, $J_0(\ell) \to 0$, $(\ell \to +\infty)$, with $J_0 = \| w*a^0 - e_0 \|_2^2$ (see 16) then

$$\varepsilon_0 = \|w*f^0 - d*e_0\|_2 \leqslant \|w*(a^0*d) - d*e_0\|_2 =$$
$$= \|d*(w*a^0 - e_0)\|_2 \leqslant \|d\|_1 \, J_0^{1/2}$$

it follows that

$$\varepsilon_0 \to 0 \qquad (\ell \to +\infty) \tag{24}$$

For every length of the filter, the value of k which realizes the minimum error is called the optimum delay for the Shaping filter of output d.

REFERENCES

[1] Claerbout,J.F.,and Robinson,E.A.,1963. *The error in least-squares inverse filtering*: Geophysics, v.29, p.118-120.

[2] Robinson,E.A., and Treitel,S.,1980, *Geophysical signal analysis*: N.J., Prentice-hall, Inc.

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