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ON WEIGHTED INEQUALITIES FOR NON STANDARD TRUNCATIONS OF SINGULAR INTEGRALS

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INTRODUCTION. The main purpose of this note is to study weighted norm inequalities for non-standard truncations of singular integrals. The non-weighted boundedness properties of the maximal operator for rectangular truncations have been studied by one of the authors in [5]. The weighted case with standard (spherical) truncation was studied by Hunt, Muckenhoupt and Wheeden [6] and Coifman and Fefferman [3]. We shall refer to the last paper for the basic properties of A_p weights.

As a basic tool in the proof of the theorem we shall use weighted inequalities for certain approximate identities, to be introduced and proved in section 2. These approximate identities are similar to those previously considered by M.Carrillo [2] and C.Calderón [2].

The proof of Theorem (1.3) is given in section 3, where some other applications are considered.

§ 1.

Let $K(x) = \dfrac{\Omega(x)}{|x|^n}$ be a Calderón-Zygmund kernel in $\mathbf{R}^n$. Here $x = (x_1,\ldots,x_n)$, $|x|$ is the euclidean length of x and Ω is a function on $\mathbf{R}^n$ homogeneous of degree zero satisfying the following two standard conditions

(1.1) Cancellation property

$$\int_{S^{n-1}} \Omega(x')dx' = 0, \text{ where } S^{n-1} \text{ is the unit sphere in } \mathbf{R}^n.$$

(1.2) Dini type condition

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty \quad , \quad \text{where} \quad \omega(\delta) = \sup_{\substack{x\varepsilon S^{n-1}\\ |h|<\delta}} |\Omega(x+h)-\Omega(x)| .$$

Let $\mathcal{F}_0$ be the family of all the balls centered at the origin of $\mathbf{R}^n$
Let $\mathcal{F}_1$ be the family of rectangles centered at the origin of $\mathbf{R}^n$ with sides parallel to the axes; i.e. $R \in \mathcal{F}_1$ if and only if $R = [-a_1,a_1] \times [-a_2,a_2] \times \dots \times [-a_n,a_n]$. Let $\mathcal{F}_2$ be the family of all rectangles centered at the origin of $\mathbf{R}^n$, $R \in \mathcal{F}_2$ if and only if R is a rotation of a rectangle in $\mathcal{F}_1$.

Associated with each one of the families $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$ we have the corresponding maximal operator

$$T_i^* f(x) = \sup_{A\in\mathcal{F}_i} |T_A f(x)| \quad , \quad \text{where} \quad T_A f(x) = \int_{y\notin A} K(y)f(x-y)dy.$$

The known results referred to in the introduction are the following:
E.Harboure (1979) [5]: T_1^* is of weak type (1,1) and T_2^* is bounded in $L^p(dx)$ for $1 < p < \infty$.
Hunt, Muckenhoupt and Wheeden (1973) [6], Coifman and Fefferman (1974) [3]: If $w \in A_1$ (i.e. $\exists\, c$: $\frac{1}{|B|}\int_B w \leq c \inf_B w$ for every ball B), then T_0^* is of weak type (1,1) with measure $w(x)dx$. If $1 < p < \infty$ and $w \in A_p$ (i.e. $\exists\, c$: $(\frac{1}{|B|}\int_B w)(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}})^{p-1} \leq c$ for every ball B then T_0^* is bounded in $L^p(wdx)$.

It is also known after a work of Kurtz and Wheeden, [7], that the Dini type condition on the kernel can not be weakened to a Hörmander type condition.
Our main result is the following theorem.

(1.3) THEOREM.
(1.4) *If* $1 < p < \infty$ *and* $w \in A_p$, *then* $\|T_2^*\|_{L^p(wdx)} \leq C\, \|f\|_{L^p(wdx)}$;

(1.5) *If* $w \in A_1$, *then* $\displaystyle\int_{\{x:T_1^*f(x)>\lambda\}} w(x)dx \leq \frac{C}{\lambda}\, \|f\|_{L^1(wdx)}$;

where C *depends only on the dimension and on the* A_p *norm of* w.

Observe that $T_2^* \geq T_1^*$, hence T_1^* is also $L^p(wdx)$ - bounded provided that $w \in A_p$ and $1 < p < \infty$.
In proving Theorem (1.3) the idea, as in the non-weighted case, is to obtain an upper estimate of T_i^* (i=1,2) by T_0^* plus the maximal

operator of certain approximate identity. Theorem (1.3) is then a consequence of weighted inequalities for approximate identities which we shall obtain in the next paragraph. We will give a detailed proof of Theorem (1.3) on § 3.

§ 2.

Let A be a bounded convex set in $\mathbf{R}^n$ with non empty interior which contains the origin. Let B be the smallest ball centered at the origin whose closure contains A. The number $e = \frac{|B|}{|A|}$ is a measure of the excentricity of A.

If k(x) is a real valued function defined on $\mathbf{R}^n$ we shall use the notation: $k_\varepsilon(x) = \frac{1}{\varepsilon^n} k(\frac{x}{\varepsilon})$ and $k_{\varepsilon,\rho}(x) = \frac{1}{\varepsilon^n} k(\frac{\rho(x)}{\varepsilon})$, where ε is a positive real number and ρ is a rotation in $\mathbf{R}^n$.

In the proof of the next lemma we shall use the following result due to E.Stein and N.Weiss [9].

LEMMA. *Let* $\{T_j\}_{j=1}^{\infty}$ *be a sequence of sublinear operators which are uniformly of weak type* (1,1). *Let* $\{c_j\}$ *be a sequence of positive numbers satisfying* $\Sigma\, c_j |\log c_j| < \infty$. *Then the operator* $\Sigma\, c_j T_j$ *is of weak type* (1,1).

(2.1) LEMMA. *Let* $k(x) \geqslant 0$ *be such that*

(2.2) $$k(x) \leqslant \sum_{j\in Z} b_j \chi_{A_j}(x) ,$$

where A_j *is a bounded convex set with non empty interior containing the origin and* $b_j > 0$. *Set* $a_j = b_j |A_j|$ *and let* e_j *be the excentricity of* A_j. *Then*

(2.3) *for* $1 < p < \infty$, *the maximal operator* $\sup_{\varepsilon,\rho} |k_{\varepsilon,\rho} * f|$ *is bounded in* $L^p(wdx)$ *for every* $w \in A_p$ *provided that the sum* $\sum_{j\in Z} a_j e_j^{1-\gamma}$ *is finite for every* $\gamma > 0$;

(2.4) *the maximal operator* $\sup_{\varepsilon>0} |k_\varepsilon * f|$ *is of weak type* (1,1) *with measure* wdx *for every* $w \in A_1$, *provided that the sum* $\sum_{j\in Z} a_j\, e_j^{1-\gamma}\, |\log(a_j e_j^{1-\gamma})|$ *is finite for every* $\gamma > 0$.

Proof. In order to prove (2.3) let us first observe that, from (2.2), we readily have

$$|k_{\varepsilon,\rho}*f(x)| \leqslant \sum_{j\in Z} b_j|A_j| \frac{1}{\varepsilon|\rho^{-1}(A_j)|}\int_{\varepsilon\rho^{-1}(A_j)}|f(x-y)|dy.$$

Hence

$$\sup_{\varepsilon,\rho}|k_{\varepsilon,\rho}*f(x)| \leqslant \sum_{j\in Z} a_j M_j f(x) \quad , \tag{2.5}$$

where $\quad M_j f(x) = \sup_{\varepsilon,\rho}\frac{1}{\varepsilon|\rho(A_j)|}\int_{\varepsilon\rho(A_j)}|f(x-y)|dy.$

Let us now study the $L^p(wdx)$ boundedness of $M_j f$. For $w \in A_p$, $1 < p < \infty$, there exists $1 < r < p$ such that $w \in A_r$ (See [3] for a proof). Let $B_j = B(0,r_j)$ be the smallest ball centered at the origin whose closure contains A_j. Applying Hölder's inequality and the A_r condition for w, we get

$$\frac{1}{\varepsilon|\rho(A_j)|}\int_{\varepsilon\rho(A_j)}|f(x-y)|dy \leqslant \frac{1}{\varepsilon^n|A_j|}\int_{B(x,\varepsilon r_j)}|f(y)|w(y)^{1/r}w(y)^{-1/r}\,dy \leqslant$$

$$\leqslant \frac{1}{\varepsilon^n|A_j|}\{\int_{B(x,\varepsilon r_j)}|f|^r w\}^{1/r}.\{\int_{B(x,\varepsilon r_j)} w^{-\frac{1}{r-1}}\}^{\frac{r-1}{r}} \leqslant$$

$$\leqslant c.e_j\,\{\frac{1}{\int_{B(x,\varepsilon r_j)} w}\int_{B(x,\varepsilon r_j)}|f|^r w\}^{1/r}.$$

Since the weighted maximal function

$$M_w g(x) = \sup_{s>0}\frac{1}{w(B(x,s))}\int_{B(x,s)}|g|\,w$$

is of weak type (1,1) with weight w, we have

$$w(\{x:M_j f(x) > \lambda\}) \leqslant C\,\frac{e_j^r}{\lambda^r}\int|f|^r\,w\,.$$

In other words, $M_j f$ is of weak type (r,r) with weight w and constant e_j^r. On the other hand, it is clear that $\|M_j f\|_{L^\infty} \leqslant \|f\|_{L^\infty}$. So that, by Marcinkiewicz interpolation theorem we obtain

$$\|M_j f\|_{L^p(wdx)} \leqslant C\,e_j^{r/p}\,\|f\|_{L^p(wdx)}\,.$$

This estimate, Minkowski's inequality and (2.5) give

$$\| \sup_{\varepsilon,\rho} |k_{\varepsilon,\rho} * f| \|_{L^p(wdx)} \leqslant C \{ \sum_{j\varepsilon Z} a_j \, e_j^{r/p} \} \, \|f\|_{L^p(wdx)} .$$

Let us now prove (2.4). As before we have

$$(2.6) \qquad \sup_{\varepsilon} |k_\varepsilon * f(x)| \leqslant \sum_{j\varepsilon Z} a_j M_j \, f(x) \ ,$$

where $$M_j f(x) = \sup_{\varepsilon>0} \frac{1}{|\varepsilon A_j|} \int_{\varepsilon A_j} |f(x-y)| \, dy .$$

Let $w \in A_1$, then there exist $\delta > 0$, $c > 0$ such that the "reverse Hölder inequality"

$$\frac{1}{|B|} \int_B w^{1+\delta} \leqslant C \Big(\frac{1}{|B|} \int_B w \Big)^{1+\delta}$$

holds for every ball B, (See [3]).

Since $w \in A_1$ we have

$$(2.7) \qquad \frac{1}{|B|} \int_B w^{1+\delta} \leqslant C \inf_B w^{1+\delta}$$

for every ball B. Consequently, applying Hölder's inequality and then (2.7), it follows that

$$\frac{1}{|\varepsilon A_j|} \int_{\varepsilon A_j} |f(x-y)| dy \leqslant \left\{ \frac{\int_{x+\varepsilon A_j} w^{1+\delta}}{|\varepsilon A_j|} \right\}^{\frac{1}{1+\delta}} \cdot \left\{ \frac{1}{\int_{x+\varepsilon A_j} w} \int_{x+\varepsilon A_j} |f(y)| dy \right\} \leqslant$$

$$\leqslant \left\{ \frac{\int_{B(x,\varepsilon r_j)} w^{1+\delta}}{|B(x,\varepsilon r_j)|} \right\}^{\frac{1}{1+\delta}} \cdot e_j^{\frac{1}{1+\delta}} \cdot \left\{ \frac{1}{\int_{x+\varepsilon A_j} w} \int_{x+\varepsilon A_j} |f(y)| \, dy \right\} \leqslant$$

$$\leqslant C . e_j^{\frac{1}{1+\delta}} \left\{ \frac{1}{\int_{x+\varepsilon A_j} w} \int_{x+\varepsilon A_j} |f| \, w \right\} .$$

From the last inequality and (2.6) we get

$$\sup_{\varepsilon} |k_\varepsilon * f| \leqslant C \sum_{j\varepsilon Z} a_j \, e_j^{\frac{1}{1+\delta}} \, M_{w,j} f(x)$$

where $$M_{w,j} f(x) = \sup_{\varepsilon} \frac{1}{\int_{x+\varepsilon A_j} w} \int_{x+\varepsilon A_j} |f| w .$$

From Besicowitch type covering lemmas (see [4]), it follows that the maximal operator $M_{w,j}$ is of weak type (1,1) with weight w and norm independent of j. Then, the result is a consequence of the "entropy" hypothesis which allows us to apply the preceding lemma.

A particular case of Lemma (2.1) gives the following weighted extension of M.T.Carrillo theorem which will be used in proving Theorem (1.3).

(2.8) COROLLARY. *Let* $k \geqslant 0$ *be non-increasing along rays. Suppose that the sets* $A_j = \{x: k(x) \geqslant 2^j\}$ *are convex and bounded for every* $j \in Z$. *Then* (2.3) *and* (2.4) *hold true with* $b_j = 2^j$.

Proof. Observe that $k(x) \cong \sum_{j\in Z} 2^j \chi_{A_j}(x)$, where the A_j's satisfy the hypotheses of Lemma (2.1).

§ 3.

In this paragraph we will give several applications of the lemma proved in §2. The first one is the proof of Theorem (1.3) on non-standard truncations of singular integrals.

(3.1) PROOF OF THEOREM (1.3). Let R be a rectangle belonging to F_i (i=1,2). We can regard R as a rotation ρ followed by a dilation ε of a rectangle R' with sides parallel to the axes whose smallest side is in the x_n-direction and has lenght 2. Set

$$S_\varepsilon = \{x \in \mathbf{R}^n: -\varepsilon \leqslant x_n \leqslant \varepsilon \quad \text{and} \quad |x| \geqslant \varepsilon\}.$$

Clearly $R' \subset B(0,1) \cup S_1$ and hence $R \subset B(0,\varepsilon) \cup \rho(S_\varepsilon)$. So that

$$|T_R f(x)| \leqslant |T_{B(0,\varepsilon)} f(x)| + (k_{\varepsilon,\rho} * |f|)(x) \ ,$$

where $$k(x) = \chi_{B(0,1)}(x) + \frac{1}{|x|^n} \chi_{S_1}(x).$$

Hence $$T_1^* f(x) \leqslant T_0^* f(x) + \sup_{1\leq i\leq n} \sup_{\varepsilon>0} (k_{\varepsilon,\rho_i} * |f|)(x) \ ,$$

where ρ_i is a rotation in $\mathbf{R}^n$ taking the hyperplane $x_i = 0$ into the hyperplane $x_n = 0$. Also

$$T_2^* f(x) \leqslant T_0^* f(x) + \sup_\rho \sup_\varepsilon (k_{\varepsilon,\rho} * |f|)(x).$$

It suffices to prove that this kernel falls into the scope of corollary (2.8). Since the hypothesis required in (2.4) is stronger than that of (2.3), we will just check the "entropy" condition

$$\sum a_j\ e_j^{1-\gamma}\ |\log(a_j e_j^{1-\gamma})| < \infty .$$

If $j > 0$, then $A_j = \{x\colon k(x) \geqslant 2^j\} = \emptyset$. If $j \leqslant 0$ the A_j's are convex and bounded sets and there exist a constant C_n, depending only on the dimension, such that

$$\{x\colon |x_n| \leqslant 1,\ |x_i| \leqslant C_n\ 2^{-j/n};\ 1 \leqslant i < n\} \subsetneq A_j \subset$$
$$\subset \{x\colon |x_n| \leqslant 1,\ |x_i| \leqslant 2^{-j/n};\ 1 \leqslant i < n\}.$$

Thus $|A_j| \leqslant 2^n . 2^{-j(1-\frac{1}{n})}$; $e_j \leqslant C_n'\ 2^{-j/n}$ and $a_j = 2^j|A_j| \leqslant C_n''\ 2^{j/n}$.

Therefore

$$\sum_{j=-1}^{-\infty} a_j e_j^{1-\gamma}|\log(a_j e_j^{1-\gamma})| \leqslant C_n \sum_{j=-1}^{-\infty} e^{\frac{j\gamma}{n}}\ |\log C_n e^{\frac{j\gamma}{n}}| ,$$

which is clearly finite.

In [1] C.Calderón studies the differentiation properties through the dilations of unbounded star-shaped sets. As a second application of the general result on section 2 we shall obtain weighted inequalities for approximate identities related to some particular shapes of those considered by C.Calderón. (See also [4] page 291).

Let K and ϕ be two non-negative non-increasing functions defined on $\mathbf{R}^+$. Suppose that ϕ is bounded above. Let $x' = (x_1, x_2, \ldots, x_{n-1})$ and $|x'|$ its length. Set $E = \{x \in \mathbf{R}^n\colon |x_n| \leqslant \phi(|x'|)\}$ and $k(x) = K(|x|)\chi_E(x)$. We shall study weighted boundedness properties of

$$Mf(x) = \sup_{\varepsilon>0} |(k_\varepsilon * f)(x)| .$$

For $\gamma \geqslant 0$ we set $\psi_\gamma(t) = K(t)\phi^\gamma(t)t^{n-1-\gamma}$. With this notation we have the following result:

(3.2) THEOREM. *If*

(3.3) $\quad \int_0^1 \psi_0(t)dt < \infty \quad$ *and*

(3.4) $\quad \int_0^\infty \psi_\gamma(t)\ |\log\psi_\gamma(t)|dt < \infty$, *for every* $\gamma > 0$;

then the maximal operator Mf *is of weighted weak type* $(1,1)$ *for* *every weight* $w \in A_1$.

Proof. We may assume without loss of generality that $\phi \leqslant 1$ and that $K \leqslant e^{-1}$ outside the unit ball. In order to apply the second part of Lemma (2.1), let us first observe that

$$k(x) \leqslant K(|x|)\chi_0(x) + \sum_{j\geq 0} K(2^j)\chi_j(x) = k_0(x)+k_\infty(x)$$

where χ_0 is the characteristic function of the ball $B(0,2)$ and χ_j is the characteristic function of the cylindrical region

$$A_j = \{x: |x'| \leqslant 2^{j+1} \quad \text{and} \quad |x_n| \leqslant \phi(2^j)\} \ ; \ j \geqslant 0.$$

The kernel $k_0(x)$ is radial and non-increasing so that we have

$$\sup_\varepsilon |(k_{0,\varepsilon} * f)(x)| \leqslant C.f^*(x) \ ,$$

where f^* is the Hardy-Littlewood maximal function and $C = \int k_0(x)dx$, which is finite by (3.3). Then, using the classical result for f^*, it remains only to study the kernel k_∞. With the notation of Lemma (2.1) for $j \geqslant 0$ we have

$$b_j = K(2^j); \ a_j = b_j|A_j| = CK(2^j)2^{j(n-1)}\phi(2^j) \ ;$$

$$e_j \cong C.\frac{2^j}{\phi(2^j)} \quad \text{and} \quad a_j e_j^{1-\gamma} = CK(2^j).\phi(2^j)^\gamma.(2^j)^{n-1-\gamma}.2^j \ ,$$

where C depends only on the dimension n. Thus

$$\sum_{j\geq 0} a_j e_j^{1-\gamma}|\log(a_j e_j^{1-\gamma})| \leqslant \sum_{j\geq 0} a_j e_j^{1-\gamma}|\log(K(2^j)\phi(2^j)^\gamma)| \ +$$

$$+ \ C' \sum_{j\geq 0} a_j e_j^{1-\gamma}(j+C) = I + II.$$

Since the function

$$K(t)\phi(t)^\gamma|\log(K(t)\phi(t)^\gamma)|$$

is non increasing for $t \geqslant 1$, then

$$\text{(3.5)} \qquad I \leqslant C\int_1^\infty K(t)\phi(t)^\gamma|\log(K(t)\phi^\gamma(t))|t^{n-1-\gamma}dt \leqslant$$

$$\leqslant C\int_1^\infty \psi_\gamma(t)|\log\psi_\gamma(t)|dt + C\int_1^\infty \psi_\gamma(t)\log t\,dt \,.$$

Also

$$\text{(3.6)} \qquad II \leqslant C\sum_{j\geq 0} K(2^j)\phi(2^j)^\gamma(2^j)^{n-1-\gamma}\log(2^{j+C'})2^j \ \leqslant$$

$$\leq C\int_1^\infty \psi_\gamma(t)\log(C't)dt.$$

The last integral on the right hand side of (3.5) and the right hand side of (3.6) are both of the same type. Let $0 < \varepsilon < \gamma$. Since $\phi \leq 1$ we have

$$\int_1^\infty \psi_\gamma(t)\log Ct\, dt \leq C\int_1^\infty \psi_\gamma(t)t^\varepsilon\, dt = C\int_1^\infty K(t)\phi^\gamma(t)t^{n-1-\gamma+\varepsilon}\, dt \leq$$

$$\leq C\int_1^\infty K(t)\phi^{\gamma-\varepsilon}(t)\ t^{n-1-(\gamma-\varepsilon)}\, dt\ ,$$

which is finite because of (3.4) with γ-ε instead of γ.

(3.7) REMARK. Given a curve of the form $x_n = \phi(x_1)$, we may generate two different types of solids of revolution, E_n and E_1, according we rotate it about the x_n or the x_1 axis. In the preceding theorem the first type of solid was used to cut the kernel K. With a similar reasoning we can get the same conclusion for E_1 under the hypotheses (3.3) and

$$(3.8)\quad \int_1^\infty K(t)\phi(t)^{\gamma(n-1)}t^{(n-1)(1-\gamma)}\,|\log K(t)\phi(t)^{\gamma(n-1)}t^{(n-1)(1-\gamma)}|dt < \infty$$

instead of (3.4). Notice that for the unweighted case, i.e. $\gamma=1$, (3.8) is weaker than (3.4) as assumption on K. However both sets of hypotheses are the same if we want to obtain the boundedness for all the A_1 - weights.

(3.9) REMARK. Similar results hold with other monotonicity properties on K and ϕ.

(3.10) EXAMPLE. If $\psi(t) \equiv 1$ and $K(t) = \frac{1}{t^n}\chi_{[1,\infty)} + \chi_{(0,1)}$,

we have $\int_1^\infty \psi_\gamma|\log\psi_\gamma| = \int_1^\infty t^{-1-\gamma}|\log t^{-1-\gamma}|\ dt < \infty$ for every γ. This means that the second part of Theorem (1.3) can also be proved using Theorem (3.2).

(3.11) EXAMPLE. If we set $K\equiv 1$, hypotheses (3.4) becomes

$$\int_1^\infty \phi^\gamma(t)t^{n-1-\gamma}|\log\phi^\gamma(t)t^{n-1-\gamma}|\ dt < \infty$$

for every $\gamma > 0$. This is satisfied if, for example, ϕ decays exponentially. Theorem (3.2) gives weighted inequalities for the maximal operator associated to differentiation through certain unbounded star-shaped sets.

Finally, we can also apply Lemma (2.1) to iterated Poisson kernels considered by Rudin [8] and M.de Guzmán [4], page 290. Let

$$k(x) = (1 + x_1^2)^{-1} \dots (1 + x_n^2)^{-1} ,$$

then

$$\sup_{\varepsilon>0} |k_\varepsilon * f|$$

is of weighted weak type (1,1) for every $w \in A_1$. The proof is a straight forward verification that the estimate given in [4] satisfies hypotheses of our lemma.

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