§ 1. INTRODUCTION

The class of symbols $S^m_{\rho,\delta}$ was introduced by Hörmander in [11] where he proved that they give $L^2$ bounded pseudo-differential operators when $m=0$ and $0 \leq \delta < \rho \leq 1$. Other continuity results within this framework were given in [12], [14], [15], [17]. Then Calderón and Vaillancourt proved ([4], [5]) that to obtain boundedness, it is enough to assume $m < 0$, $0 \leq \delta < \rho \leq 1$, $\delta < 1$ (these conditions are necessary for $L^2$ continuity ([8],[12])) and this improvement had a remarkable application to local solvability [3].

The next step was to strive for minimizing the number of derivatives of the symbol needed to control the norm of the operator [10], [13], [16].

In [9] Coifman and Meyer developed a systematic approach to study boundedness of pseudo-differential operators, proving among a number of results, the following

THEOREM 1. Let $n \geq 1$ be an integer and set $N = \lceil n/2 \rceil + 1$. Assume that $a(x,\xi)$ and its derivatives $D^\alpha D^\beta_{\xi_{\xi}}a(x,\xi)$, $|\alpha|, |\beta| \leq N$ are continuous in $\mathbb{R}^n \times \mathbb{R}^n$ and satisfy

\begin{equation}
|D^\alpha D^\beta_{\xi_{\xi}}a(x,\xi)| \leq C(1+|\xi|)^{\delta(|\alpha|-|\beta|)} , \quad x, \xi \in \mathbb{R}^n ,
\end{equation}

where $0 < \delta < 1$ and $C > 0$ are two constants. Then the operator

\begin{equation}
a(x,D)u(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} a(x,\xi) \hat{u}(\xi) \, d\xi
\end{equation}

is bounded in $L^2(\mathbb{R}^n)$.

Theorem 1 is optimal in the sense that $N = \lceil n/2 \rceil + 1$ cannot be replaced by a smaller integer. The symbol

\[ a(x,\xi) = e^{ix \cdot \xi}(1+|\xi|^2)^{-n/4}e^{-|x|^2} \]

satisfies (1.1) with $\delta = 0$ for
|a| \leq n/2 and all $\beta$ and yet $a(x,D)$ is not bounded in $L^2(\mathbb{R}^n)$.

In this work we fill the gap between $n/2$ and $[n/2]+1$ by considering Hölder classes of symbols. If we denote by $S^0_{\delta,\delta}(N)$, $N \in \mathbb{Z}^+$, the space of symbols that satisfy (1.1) for $|\alpha|, |\beta| \leq N$, theorem 1 can be expressed as: $N > n/2$ implies that $a(x,D)$ is bounded when $a \in S^0_{\delta,\delta}(N)$. Considering Hölder classes we may define $S^m_{\rho,\delta}(N)$ for any real $N > 0$ (precise definitions are given in §2). We then have

**Theorem 2.** Let $a(x,\xi)$ belong to $S^m_{\rho,\delta}(N)$, $x, \xi \in \mathbb{R}^n$. If $m, \rho, N$ are real numbers satisfying $m < 0$, $N > \frac{n}{2}$, $0 < \delta < \rho < 1$, and $a(x,D)$ is given by (1.2) then $a(x,D)$ is bounded in $L^2(\mathbb{R}^n)$.

The proof of theorem 2 uses the techniques of Coifman and Meyer, in particular, the almost orthogonality principle in the sharp form given by Alvarez Alonso and Calderón [1], [2]. The paper is organized as follows: in §2 we define the appropriate classes of symbols (related classes appear in [6] and [7]), in §3 we prove some technical lemmas, in §4 we prove theorem 2 and in §5 we discuss the necessity of the regularity hypotheses of theorem 2.

§ 2. CLASSES OF SYMBOLS

Consider a function $a(x,\xi)$ in $\mathbb{R}^n \times \mathbb{R}^n$. We define the partial finite differences in $x$ and $\xi$ by

\[
\begin{align*}
d^1_y a(x,\xi) &= a(x+y,\xi) - a(x,\xi) \\
d^2_\eta a(x,\xi) &= a(x,\xi+\eta) - a(x,\eta).
\end{align*}
\]

Then we define for $0 \leq \varepsilon \leq 1$,

\[
\begin{align*}
\Delta^\varepsilon_x a(x,\xi) &= \sup_{y \neq 0} |y|^{-\varepsilon} |d^1_y a(x,\xi)| \\
\Delta^\varepsilon_\xi a(x,\xi) &= \sup_{\eta \neq 0} |\eta|^{-\varepsilon} |d^2_\eta a(x,\xi)| \\
\Delta^\varepsilon_{x,\xi} a(x,\xi) &= \sup_{y, \eta \neq 0} |y|^{-\varepsilon} |\eta|^{-\varepsilon} |d^1_y d^2_\eta a(x,\xi)|.
\end{align*}
\]

Let $m, \rho, \delta$ be real numbers satisfying $0 \leq \rho < 1$, $0 \leq \delta < 1$. If $k$ is a non-negative integer and $N = k + \varepsilon$, $0 \leq \varepsilon < 1$, we denote by $S^m_{\rho,\delta}(N)$
the space of measurable functions \(a(x, \xi)\) defined in \(\mathbb{R}^n \times \mathbb{R}^n\) such that its weak derivatives \(D_x^\alpha D_\xi^\beta a(x, \xi)\), of order \(\alpha, \beta \in \mathbb{N}_0\), \(|\alpha| = \alpha_1 + \ldots + \alpha_n \leq k\), \(|\beta| \leq k\), are locally integrable functions which satisfy for almost every \(x, \xi\) the following estimates

\[
(2.1) \quad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha_1, \beta_1}(1 + |\xi|)^{m + \delta} |\alpha| - \rho |\beta|
\]

\[
(2.2) \quad \Delta_x^\alpha D_x^\beta D_\xi^\gamma a(x, \xi) \leq C_{\alpha_1, \beta_1}(1 + |\xi|)^{m + \delta} |\alpha| + \rho (|\beta| + \epsilon)
\]

\[
(2.3) \quad \Delta_\xi^\alpha D_x^\beta D_\xi^\gamma a(x, \xi) \leq C_{\alpha_1, \beta_1}(1 + |\xi|)^{m + \delta} |\alpha| - \rho (|\beta| + \epsilon)
\]

\[
(2.4) \quad \Delta_\xi^\alpha D_x^\beta D_\xi^\gamma a(x, \xi) \leq C_{\alpha_1, \beta_1}(1 + |\xi|)^{m + \delta} |\alpha| + \rho (|\beta| + \epsilon)
\]

Notice that (2.2), (2.3) and (2.4) are superfluous if \(\epsilon = 0\). The sum of the best constants \(C_{\alpha_1, \beta_1}, C_{\alpha_2, \beta_2}, C_{\alpha_3, \beta_3}, C_{\alpha_4, \beta_4}\) that appear in (2.1), ..., (2.4) is a norm that turns \(S^m_{\rho, \delta}(N)\) into a Banach space and will be denoted by \(\| \cdot \|_{S^m_{\rho, \delta}(N)}\).

**Proposition 2.1.** \(S^m_{\rho, \delta}(N)\) is a Banach space. This space increases if \(m\) and \(\delta\) increase and \(\rho\) and \(N\) decrease. If \(a \in S^m_{\rho, \delta}(N)\) and \(|\alpha|, |\beta| < N\), it follows that

\[
D_x^\alpha D_\xi^\beta a \in S^{m-p, \delta}_{\rho, \delta}(N-\max(|\alpha|, |\beta|))
\]

and if \(b \in S^m_{\rho, \delta}(N')\), it follows that \(ab \in S^{m+m'}_{\rho, \delta}(N')\).

We now indicate the proof of \(S^m_{\rho, \delta}(N) \subseteq S^m_{\rho, \delta}(N')\) when \(N > N'\). Assume first that \(N = \epsilon\), \(N' = \epsilon'\), \(0 \leq \epsilon' < \epsilon < 1\). Then, if \(a \in S^m_{\rho, \delta}(\epsilon)\) we have, for instance, \(|a(x, \xi)| \leq C(1 + |\xi|)^m\), \(|D_y a(x, \xi)| \leq C|y|^{\epsilon(1 + |\xi|)}^{m + \delta \epsilon}\). These estimates imply that

\[
|y|^{-\epsilon'}|D_y a(x, \xi)| \leq 2C(1 + |\xi|)^m \min(|y|^{-\epsilon'}, (1 + |\xi|)^{\delta \epsilon}|y|^{\epsilon - \epsilon'})
\]

The function \(f(r) = \min(r^{-\epsilon'}, Ar^{-\epsilon'})\), \(r > 0\), \(A > 0\), has a maximum at \(r_o = A^{-1/\epsilon}\) equal to \(f(r_o) = A^{\epsilon'/\epsilon}\). It follows that

\[
|y|^{-\epsilon'}|D_y a(x, \xi)| \leq 2C(1 + |\xi|)^{m + \delta \epsilon}|y|^{\epsilon - \epsilon'}
\] so \(\Delta_x^\alpha a(x, \xi) \leq 2C(1 + |\xi|)^{m + \delta \epsilon}\). The other estimates follow in a similar fashion and we obtain \(S^m_{\rho, \delta}(\epsilon) \subseteq S^m_{\rho, \delta}(\epsilon')\).
If \( a \in S^m_\rho \delta(1) \), the estimate \(|D^\beta_x a(x, \xi)| \leq C(1+|\xi|)^{m-\rho}, |\beta| = 1\), together with the mean value theorem yield

\[(2.5) \quad |d^2_{\eta} a(x, \xi)| \leq C(1+|\xi|)^{m-\rho}|\eta|, \quad |\eta| \leq |\xi|+1.\]

On the other hand, from the triangular inequality

\[(2.6) \quad |d^2_{\eta} a(x, \xi)| \leq C'(1+|\xi|)^m, \quad \eta \in \mathbb{R}^n.\]

Thus, (2.5) and (2.6) imply

\[\Delta^2_x a(x, \xi) \leq C''(1+|\xi|)^{m-\rho},\]

as \( \rho \leq 1. \) Using this estimate and \(|a(x, \xi)| \leq C(1+|\xi|)^m\) we get

\[|d^2_{\eta} a(x, \xi)| \leq \text{const.}(1+|\xi|)^{m-\rho}|\eta|^{\epsilon}.\]

Similarly, we get the other estimates required to show that \( S^m_\rho, \delta(1) \subseteq S^m_\rho, \delta(\epsilon) \). It follows now inductively that

\[S^m_\rho, \delta(k+1) \subseteq S^m_\rho, \delta(k+\epsilon) \subseteq S^m_\rho, \delta(k+\epsilon') \subseteq S^m_\rho, \delta(k), \quad k \in \mathbb{N}, \quad 0 \leq \epsilon' < \epsilon < 1.\]

In the next section we will consider the space of Hölder functions \( \Lambda_{r}(\mathbb{R}^n) \). Let us recall some well known facts. If \( r=0 \), \( \Lambda_{o} = L^\infty(\mathbb{R}^n) \), if \( 0 < r < 1 \), \( \Lambda_{r} \) is the subspace of \( \Lambda_{o} \) of the functions satisfying

\[|f(x)-f(y)| \leq C|x-y|^r \text{ a.e., the class of } f \text{ contains a continuous representative. For general } r > 0, \text{ we write } r = [r]+r-[r] = k+\epsilon, \quad k \in \mathbb{N}, \quad 0 \leq \epsilon < 1, \text{ and } \Lambda^r \text{ is the space of the functions } f \in \Lambda_{o} \text{ with weak derivatives } \partial^a f \in \Lambda^\epsilon \text{ for } |a| \leq k. \]

When \( 0 < r < 1 \), the norm \( \|f\|_r \) is the maximum between \( \|f\|_\infty \) and the essential supremum of the quotients \(|f(x)-f(y)|/|x-y|^r \). When \( r = k+\epsilon, \quad 0 \leq \epsilon < 1, \quad k \in \mathbb{N}, \)

\[\|f\|_r = \max_{|a| \leq k} \|\partial^a f\|_r.\]

§ 3. BASIC LEMMAS

The following is a discrete version of a lemma of Alvarez-Calderón ([1], [2]) and may be refered to as the sharp almost-orthogonality principle. We include the proof for completeness.

LEMMA 3.1. Let \( s > n/2 \) and set \( r = 1-n/2s \). Then, there is a positive constant \( C = C(s, n) \) such that for any finite number of functions \( f_k \in H^s, \quad k \in \mathbb{Z}^n \), we have

\[\]
(3.1) \[ \sum_{k} \| e_k f_k \|_o^2 < C(\| \sum_{k} f_k \|_o^2)^{1-r} \] .

Here, \( H^s \) indicates the Sobolev space in \( \mathbb{R}^n \) with norm
\[ \| f \|_o^2 = (2\pi)^{-n} \int |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi , \quad \hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) dx , \]
and \( e_k \) indicates the operator of multiplication by the bounded function \( \exp(ik \cdot x) \), \( i = \sqrt{-1} \), \( x \cdot k = x_1 k_1 + \ldots + x_n k_n \).

Proof. If
\[ \omega_\lambda(\xi) = \sum_{k \in \mathbb{Z}^n} (1+|\xi-k|^{2s})^{-1} , \quad \text{for } 2s > n , \]
there is a positive constant \( C = C(s,n) \) such that \( \omega_\lambda(\xi) < C \lambda^{-n/2s} \)
for \( \xi \in \mathbb{R}^n \) and \( 0 < \lambda < 1 \). Then, by Parseval's formula
\[ \sum_{k} \| e_k f_k \|_o^2 = (2\pi)^{-n} \int |\sum_k \hat{e_k}(\xi-k)|^2 d\xi < \]
\[ < (2\pi)^{-n} \int |\sum_k (1+|\xi-k|^{2s}) \hat{e_k}(\xi-k)|^2 \omega_\lambda(\xi) d\xi < \]
\[ < C \lambda^{-n/2s} (\sum_{k} \| f_k \|_o^2 + \lambda \sum_{k} \| f_k \|_s^2) . \]

It is enough to take
\[ \lambda = \frac{\sum_{k} \| f_k \|_o^2}{\sum_{k} \| f_k \|_s^2} \]
to obtain (3.1).

Let \( k \) be a non-negative integer, \( \varepsilon \) a real number, \( 0 < \varepsilon < 1 \), and set \( s = k+\varepsilon \). It is well known that an equivalent norm for the space \( H^s \) is given by
\[ (3.2) \quad \| f \|_s^2 = \sum_{|\alpha| \leq k} \| D^\alpha f \|_o^2 + \sum_{|\alpha| = k} \int \| D^\alpha(f_t f) \|_o^2 |t|^{-n-2\varepsilon} dt \]
where \( f_t(x) = f(x+t) \).

**Lemma 3.2.** Let \( s, N \) be real numbers \( n/2 < s < N \) and consider a symbol \( a(x,\xi) \in S_{00}^0(N) \), \( x, \xi \in \mathbb{R}^n \), such that \( a(x,\xi) = 0 \) if \( |\xi| > \sqrt{n} \). Then there exists a constant \( C = C(N,s,n) \) such that
\[ (3.3) \quad \| a(x,D) \|_{L^2(L^2)} \leq C \sup_{x} \| a(x,\cdot) \|_N \]
(3.4) \[ \|a(x,D)\|_{L(L^2, H^s)} \leq C \|a\|_{S_0^0(N)}. \]

(The norm \( \| \cdot \|_N \) was defined at the end of §2).

**Proof.** It is enough to prove the lemma when \( s = k + \epsilon' \), \( N = k + \epsilon \), \( 0 < \epsilon < \epsilon' < 1 \), \( k \in \mathbb{Z}^+. \) Setting

\[ k(x,y) = (2\pi)^{-n} e^{ix \cdot \xi} a(x, \xi) d\xi, \quad \omega(y) = (1 + |y|^2)^{-s} \]

we have for \( f \in S \)

\[ |a(x,D)f(x)|^2 = \int |k(x,y)f(x-y)dy|^2 \leq \int |k(x,y)||2\omega^{-1}(y)dy| |\omega(f)|^2(x) \]

\[ \leq C \|a(x,.)\|_s^2 (\omega(f)|^2)(x). \]

Integrating both sides of this estimate we get

(3.5) \[ \|a(x,D)\|_{L(L^2)} \leq C \sup_x \|a(x,.)\|_s. \]

Using (3.2) and the fact that \( a(x,\xi) \) vanishes for \( |\xi| > \sqrt{n} \), we can estimate \( \|a(x,.)\|_s \) by \( \|a(x,.)\|_N \). This gives (3.3).

Set \( g(x) = a(x,D)f(x) \), \( f \in S \). For \( |a| \leq k \) we may write

(3.6) \[ D^\alpha g(x) = a^\alpha(x,D)f(x) \]

(3.7) \[ D^\alpha (g(x+t)-g(x)) = a^\alpha_t(x,D)f(x) \]

with

\[ a^\alpha(x,\xi) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!} (\alpha-\beta)!^{-1} \xi^\beta a(x,\xi) \]

Taking account of (3.2), and (3.6), we get

(3.8) \[ \|g\|_s^2 \leq C \left( \sum_{|\alpha| \leq k} \|a^\alpha(x,D)\|_{L(L^2)}^2 \right) + \]

\[ + \sum_{|\alpha| = k} \int_{|t| \leq 1} \frac{\|a^\alpha_t(x,D)\|_{L(L^2)}^2 |t|^{-n-2\epsilon} dt}{\|f\|_0}. \]

Thus, (3.4) follows from (3.8) and the next lemma.

**Lemma 3.3.** With notation (3.7),
\[ \|a_\alpha(x,D)\|_{L^2} \leq C \|a\|_{S^0_{\infty}(N)}^s \]
\[ \|a_\alpha(x,D)\|_{L^2} \leq C|t|^{\varepsilon'}\|a\|_{S^0_{\infty}(N)} \]

**Proof.** By (3.3), it is enough to estimate \( \|a_\alpha(x,.)\|_N \) and \( \|a_\alpha(x,.)\|_N \). This is easily done using the following

**Proposition 3.1.** Let \( f(x,\xi) \in S^0_{\infty}(\varepsilon') \), \( 0 < \varepsilon' < 1 \) and assume that \( f(x,\xi) = 0 \) if \( |\xi| > \sqrt{n} \) and set
\[ f^t(x,\xi) = e^{it.\xi}f(x+t,\xi) - f(x,\xi) \]

Then, there is a positive constant \( C = C(n) \) such that
\[ |f^t(x,\xi)| \leq C(n)(\Delta_x^{\varepsilon'} f(x,\xi) + |f(x,\xi)|)|t|^{\varepsilon'} \]
\[ |\Delta_\eta^{2\varepsilon} f^t(x,\xi)| \leq C(n)(\Delta_x^{\varepsilon'} f(x,\xi) + \Delta_\xi^{\varepsilon'} f(x,\xi) + f(x+t,\xi))|t|^{\varepsilon'}|\eta|^{\varepsilon'}. \]

**Proof.** We prove (3.10), the proof of (3.9) is simpler. It is easy to check that
\[ \Delta_\eta^{2\varepsilon} f^t(x,\xi) = e^{it.(\xi+\eta)}d_\eta^{1\varepsilon} f(x,\xi) + e^{it.(\xi+\eta)-1}d_\eta^{2\varepsilon} f(x,\xi) + \]
\[ + e^{it.\xi}(e^{it.\eta}-1)f(x+t,\xi) \]

(the difference operators \( d_\xi^1, d_\eta^2 \) were defined in §2). Thus (3.10) follows from the trivial estimate \( |e^{it-1}| \leq \min(|t|,2) \), \( t \in \mathbb{R} \).

**Lemma 3.4.** Let \( s, N \) be real numbers, \( n/2 < s < N \), and consider a symbol \( a \in S^0_{\infty}(N) \) such that \( a(x,\xi) = 0 \) if \( |\xi| \) is large enough.

Then there exists a positive constant \( C = C(N,s,n) \) such that
\[ \|a(x,D)\|_{L^2} \leq C(\sup_N \|a(.,.)\|)^r\|a\|_{S^0_{\infty}(N)}, \]

where \( r = 1-n/2s. \)

**Proof.** Set \( g = a(x,D)f, f \in S \), and consider a function \( \phi \in C^\infty_c(\mathbb{R}^n) \)
supported in \( |\xi| < \sqrt{n} \) such that
\[ \sum_{k \in \mathbb{Z}^n} \phi_k^2(\xi-k) = 1. \]

Then \( g \) can be written as a finite sum,
where
\[ \hat{f}_k(\xi) = \phi_k(\xi) \hat{f}(\xi), \quad a_k(x, \xi) = a(x, \xi + k) \phi(\xi). \]

In particular, it follows from Lemma 3.2, that
\[
\|g_k\|_2^2 \leq C \sup_x \|a(x, \cdot)\|_N^2 f_k \|_2^2
\]
\[
\|g_k\|_s^2 \leq C \|a\|_{S_0}^2 \|f_k\|_s^2.
\]

Applying Lemma (3.1) to \( g \) and observing that \( \sum \|f_k\|_s^2 = \|f\|_s^2 \) (3.12) follows.

§ 4. PROOF OF THEOREM 2

Since \( S_m^\rho, \delta \) increases with \( m \) and \( \delta \), we may assume that \( m=0 \) and \( \delta=\rho \).

There is no loss of generality in assuming that \( a(x, \xi) \) vanishes if \( |\xi| < 1 \) and we do so. Choose a non-negative function \( \phi \in C^\infty_c(\mathbb{R}^n) \)
supported in \( 1/3 < |\xi| < 1 \) and such that \( \sum_{j=0}^\infty \phi(2^{-j}\xi) = 1 \) if
\( |\xi| \geq 1/2 \). The dyadic decomposition of \( a(x, \xi) \) is

\[ a(x, \xi) = \sum_{j=0}^\infty \phi(2^{-j}\xi) \sum_{j=0}^\infty a_j(x, \xi). \]

Since \( |\xi| \sim 2^j \) when \( (x, \xi) \) is in the support of \( a_j \) we get with
\( N = k+\epsilon = \lceil \frac{n}{2} \rceil + \epsilon, \quad 0 < \epsilon < 1 \),

\[
\Delta_x^\epsilon \Delta^\alpha_x \Delta^\beta_x a_j(x, \xi) \leq C 2^{j\delta} (|\alpha| + |\beta|)
\]
\[
\Delta^\epsilon_x \Delta^\alpha_x \Delta^\beta_x a_j(x, \xi) \leq C 2^{j\delta} (|\alpha| + |\beta|)
\]
\[
\Delta^\epsilon_x \Delta^\alpha_x \Delta^\beta_x a_j(x, \xi) \leq C 2^{j\delta} (|\alpha| + |\beta|)
\]
\[
\Delta^\epsilon_x \Delta^\alpha_x \Delta^\beta_x a_j(x, \xi) \leq C 2^{j\delta} (|\alpha| + |\beta|)
\]

Let \( \psi \geq 0 \in S \) be such that \( \hat{\psi}(\xi) = 0 \) if \( |\xi| \leq 2^{-4} \) and \( \hat{\psi}(\xi) = 0 \) if
\( |\xi| \geq 2^{-3} \) and set
\[ p_j(x, \xi) = \int a_j(x-y, \xi) \psi(2^j y) 2^n dy \]
\[ q_j(x, \xi) = \int (a_j(x-y, \xi) - a_j(x, \xi)) \psi(2^j y) 2^{nj} dy \]
\[ a_j = p_j * q_j. \]

Since \( \int \psi = 1 \) it is clear that \( p_j \) satisfies estimates (4.2) with the same constant \( C \). Therefore, if we set \( \tilde{p}_j(x, \xi) = p_j(z^{-\delta_j}x, z^{\delta_j}\xi) \)
we obtain from (4.2)
\[ \| \tilde{p}_j \|_{S_0^0(N)}^o \leq C \]
\[ \| \tilde{p}_j(x, \xi) \|_N \leq C \]
with \( C \) independent of \( j \). Applying Lemma (3.4) we conclude that
\[ \| \tilde{p}_j(x, D) \|_{L^2(L^2)} \]
is uniformly bounded in \( j \), and observing that
\[ \| \tilde{p}_j(x, D) \|_{L^2(L^2)} = \| p_j(x, D) \|_{L^2(L^2)}, \]
we get
\[ \| p_j(x, D) \|_{L^2(L^2)} \leq C. \]
On the other hand, it is easy to check
that if for any \( f \in S \), we set \( g_j(x) = p_j(x, D)f(x) \) then \( \hat{g}_j(x) \) and \( \hat{h}_j(x) \) are supported in the annulus \( z^{j-2} < |\xi| < z^{j+1} \), where \( \hat{u} \) indicates the Fourier transform of \( u \) and \( p^*(x, D) \) is the adjoint of \( p(x, D) \). In particular,
\[ p_j(x, D)p_k^*(x, D) = 0 \text{ if } |j-k| > 3. \]
So we get
\[ \| \sum_{j=0}^M p_j(x, D) \|_{L^2(L^2)} \leq C, \text{ } M \in \mathbb{Z}. \]
For the symbols \( \tilde{q}_j(x, \xi) = q_j(z^{-\delta_j}x, z^{\delta_j}\xi) \) we obtain
\[ \| \tilde{q}_j \|_{S_0^0(N)}^o \leq C \]
\[ \| \tilde{q}_j(x, \xi) \|_N \leq C 2^{(\delta-1)\xi j} \]
Estimate (4.6) is obtained as (4.3). To prove (4.7) observe that
\[ |p^\xi q_j(x, \xi)| = |\int_{-y}^1 p^\xi a_j(x, \xi) \psi(2^j y) 2^{nj} dy| \leq \]
\[ \leq C z^j(|\xi| - |\beta|) |y|^\xi \psi(2^j y) 2^{nj} dy = C z^j(\xi(\delta-1) - |\beta| \delta|) \]

\[ \text{where } z = \sqrt{\xi \delta}. \]
Analogously,

\[ |d^2_{n} \Delta_{q_{j}}(x,\xi)| = \int |d_{n} d_{y} \Delta_{q_{j}}(x,\xi) \psi(2^{j} y) z^{n} dy| \leq \]

\[ \leq C 2^{j} |\varepsilon(\delta-1) - (|\beta| + \varepsilon)| |\eta|^{\varepsilon} , \quad |\beta| < k. \]

The above estimates imply (4.7). Using (4.6), (4.7) and Lemma 3.4 we obtain

\[ \|q_{j}(x,D)\|_{L(L^{2})} = \|\tilde{q}_{j}(x,D)\|_{L(L^{2})} \leq C 2^{j\varepsilon(\delta-1)(1-n/2s)} \]

Thus \( \|q_{j}(x,D)\|_{L(L^{2})} \) is dominated by a geometric convergent series, and together with (4.5), this implies

\[ \| \sum_{j=0}^{\infty} a_{j}(x,D) \|_{L(L^{2})} \leq C. \]

Since \( \sum_{j=0}^{\infty} a_{j}(x,D)f(x) \) converges to \( a(x,D)f \) in \( S' \) the proof is complete.

§ 5. NECESSARY CONDITIONS OF REGULARITY

In this section we consider separate regularity in the variables \( x \) and \( \xi \). If \( N = k+\varepsilon, N' = k'+\varepsilon' \), \( k, k' \in \mathbb{N} \), \( 0 \leq \varepsilon, \varepsilon' < 1 \), we define \( S^{m}_{\rho,\delta}(N,N') \) by the following estimates, valid for \( |a| \leq k, \]

\[ |D^{m}_{x} D^{n}_{\xi} a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta} |a|-\rho|\beta| \]

\[ \Delta^{m}_{x} D^{n}_{\xi} a(x,\xi) \leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta} |a|-\rho|\beta| \]

\[ \Delta^{m}_{x} D^{n}_{\xi} a(x,\xi) \leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta} |a|-\rho|\beta| + \rho'| \]

\[ \Delta^{m}_{x,\xi} D^{n}_{\xi} a(x,\xi) \leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta} |a|-\rho|\beta| + \rho'| \]

where we have used the notation of § 2 and \( \Delta^{m}_{x,\xi} a(x,\xi) \) indicates the essential supremum of \( |y|^{-\rho} |\eta|^{-\rho'} |d_{y} d_{\eta} a(x,\xi)| \), \( y, \eta \in \mathbb{R}^{n} \). We indicate with \( S_{-n}(N,N') \) the intersection \( \cap_{m} S^{m}_{\rho,\delta}(N,N') \) with the projective limit topology. In the same way we may define

\[ S_{\rho,\delta}(0,\infty) \]

The example of Coifman and Meyer ([9]) \( a(x,\xi) = (1+|\xi|^{2})^{-n/4} e^{ix.\xi} - |x|^{2} \)

in \( \mathbb{R}^{n} \times \mathbb{R}^{n} \), exhibits a symbol in \( S^{m}_{\rho,\delta}(\mathbb{R}^{n},\mathbb{R}^{n}) \) for which \( a(x,D) \) is
unbounded in $L^2$, showing that lack of regularity in $x$ cannot be
compensated for with high regularity in $\xi$. In this section we pro-
ve

**THEOREM 3.** Assume that $a(x,D)$ is $L^2$-bounded for all $a(x,\xi)$ in
$S^{-\infty}(\infty,N)$. Then $N \geq \frac{n}{2}$.

Observe that Theorem 2 shows that all symbols in $S^d(N,N')$ yield
bounded operators if $N, N' > n/2$. We do not know if all symbols in
$S^d(\infty,\frac{n}{2})$ give bounded operators.

Let us denote by $L(N)$ the closed subspace of $S^d_{oo}(\infty,N)$ of those
symbols vanishing for $|\xi| > \sqrt{n}$. Theorem 3 follows from

**LEMMA 5.1.** Assume $a(x,D)$ is $L^2$-bounded for all $a(x,\xi)$ in $L(N)$.
Then $N \geq n/2$.

**Proof.** We will consider symbols given by sums of exponentials as in
[8] and [12]. By the closed graph theorem there is a continuous
seminorm $p$ in $S^d_{oo}(\infty,N)$ such that

$$(5.1) \quad \|a(x,D)\|_L \leq p(a), \quad a \in L(N).$$

Take $\phi \in C_c^\infty(\mathbb{R}^k)$, equal to one in the cube $\max|\xi_1| < 1/4$ and vanis-
hing outside the cube $\max|\xi_1| < 1/2$.

For any positive integer $\lambda$, set

$$(5.2) \quad a_\lambda(x,\xi) = \sum_{\alpha \in \Lambda_\lambda} e^{-i\lambda^{-1}|x|_2^\lambda} N(\lambda \xi - \alpha)$$

where $A_\lambda$ is the set of non-negative multi-indices $\alpha \in N^+$ such that
$max \alpha_1 < \lambda - 1$. In particular, the cardinal of $A_\lambda$ is $\lambda^{2n}$ and $a_\lambda(x,\xi)$
vanishes if $\max|\xi_1| > 1$. The terms in (5.2) have disjoint supports
and it is a simple exercise in H"older functions to show that if $p$ is
a continuous seminorm in $S^d_{oo}(\infty,N)$,

$$(5.3) \quad p(a_\lambda) \leq C, \quad \lambda = 1,2,...$$

To estimate the norm of $a_\lambda(x,D)$, take $f_o \in S$, $\|f_o\|_o = 1$, so that
$
\hat{f}_o$ is supported in the cube $\max|\xi_1| < 1/4$ and set

$$(5.4) \quad \hat{\hat{f}}(\xi) = \sum_{\alpha \in \Lambda_\lambda} \hat{f}_o(\lambda \xi - \alpha).$$
As the terms are orthogonal,
\[ \| f \|_o^2 = \sum_{\alpha \in \mathcal{A}_\lambda} \lambda^{-n} \| f_\alpha \|_o^2 = 1. \]

On the other hand, since \( \hat{f}_o = \hat{f}_o \),
\[ a(x,\xi) \hat{f}(\xi) = \sum_{\alpha \in \mathcal{A}_\lambda} e^{-i\lambda^{-1}a \cdot x,\xi} \hat{f}_o(\lambda \xi - \alpha), \]
so
\[ g(x) = a(x,D)f(x) = (2\pi)^{-n} \lambda^{-n-N} \int e^{ix \cdot \xi} \hat{f}_o(\lambda \xi) d\xi \]
and
\[ (5.4) \quad \| a_\lambda(x,D) \|_{L(L^2)} \leq \| g \|_o = \lambda^{-n-N}. \]

It follows from (5.1), (5.3) and (5.4) that \( \lambda^{-n-N} \) is bounded for \( \lambda = 1, 2, \ldots \), so \( n-2N \leq 0 \).

REFERENCES


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