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A CLASS OF BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

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§ 1. INTRODUCTION

The class of symbols $S^m_{\rho,\delta}$ was introduced by Hörmander in [11] where he proved that they give L^2 bounded pseudo-differential operators when m=0 and $0 \leqslant \delta < \rho \leqslant 1$. Other continuity results within this framework were given in [12], [14], [15], [17]. Then Calderón and Vaillancourt proved ([4],[5]) that to obtain boundedness, it is enough to assume $m \leqslant 0$, $0 \leqslant \delta \leqslant \rho \leqslant 1$, $\delta < 1$ (these conditions are necessary for L^2 continuity ([8],[12])) and this improvement had a remarkable application to local solvability [3].

The next step was to strive for minimizing the number of derivatives of the symbol needed to control the norm of the operator [10], [13], [16].

In [9] Coifman and Meyer developed a systematic approach to study boundedness of pseudo-differential operators, proving among a number of results, the following

THEOREM 1. *Let* $n \geqslant 1$ *be an integer and set* $N = [n/2]+1$. *Assume that* $a(x,\xi)$ *and its derivatives* $D^\alpha_x D^\beta_\xi a(x,\xi)$, $|\alpha|$, $|\beta| \leqslant N$ *are continuous in* $\mathbf{R}^n \times \mathbf{R}^n$ *and satisfy*

$$(1.1) \qquad |D^\alpha_x D^\beta_\xi a(x,\xi)| \leqslant C(1+|\xi|)^{\delta(|\alpha|-|\beta|)} \quad , \quad x,\xi \in \mathbf{R}^n \ ,$$

where $0 \leqslant \delta < 1$ *and* $C > 0$ *are two constants. Then the operator*

$$(1.2) \qquad a(x,D)u(x) = (2\pi)^{-n} \int e^{ix.\xi} a(x,\xi)\hat{u}(\xi)\, d\xi$$

is bounded in $L^2(\mathbf{R}^n)$.

Theorem 1 is optimal in the sense that $N = [n/2]+1$ cannot be replaced by a smaller *integer*. The symbol
$a(x,\xi) = e^{ix.\xi}(1+|\xi|^2)^{-n/4}e^{-|x|^2}$ satisfies (1.1) with $\delta = 0$ for

$|\alpha| \leqslant n/2$ and all β and yet $a(x,D)$ is not bounded in L^2([9]).
In this work we fill the gap between $n/2$ and $[n/2]+1$ by considering Hölder classes of symbols. If we denote by $S^o_{\delta,\delta}(N)$, $N \in \mathbf{Z}^+$, the space of symbols that satisfy (1.1) for $|\alpha|$, $|\beta| \leqslant N$, theorem 1 can be expressed as: $N > n/2$ *implies that* $a(x,D)$ *is bounded when* $a \in S^o_{\delta,\delta}(N)$. Considering Hölder classes we may define $S^m_{\rho,\delta}(N)$ for any *real* $N > 0$ (precise definitions are given in §2). We then have

THEOREM 2. *Let* $a(x,\xi)$ *belong to* $S^m_{\rho,\delta}(N)$, $x,\xi \in \mathbf{R}^n$. *If* m,δ,ρ,N *are real numbers satisfying* $m \leqslant 0$, $N > \frac{n}{2}$, $0 \leqslant \delta \leqslant \rho \leqslant 1$, $\delta < 1$ *and* $a(x,D)$ *is given by* (1.2) *then* $a(x,D)$ *is bounded in* $L^2(\mathbf{R}^n)$.

The proof of theorem 2 uses the techniques of Coifman and Meyer, in particular, the almost orthogonality principle in the sharp form given by Alvarez Alonso and Calderón [1], [2]. The paper is organized as follows: in §2 we define the appropriate classes of symbols (related classes appear in [6] and [7]), in §3 we prove some technical lemmas, in §4 we prove theorem 2 and in §5 we discuss the necessity of the regularity hypotheses of theorem 2.

§ 2. CLASSES OF SYMBOLS

Consider a function $a(x,\xi)$ in $\mathbf{R}^n \times \mathbf{R}^n$. We define the partial finite differences in x and ξ by

$$d^1_y a(x,\xi) = a(x+y,\xi) - a(x,\xi)$$

$$d^2_\eta a(x,\xi) = a(x,\xi+\eta) - a(x,\eta).$$

Then we define for $0 \leqslant \varepsilon \leqslant 1$,

$$\Delta^\varepsilon_x a(x,\xi) = \sup_{y \neq 0} |y|^{-\varepsilon} |d^1_y a(x,\xi)|$$

$$\Delta^\varepsilon_\xi a(x,\xi) = \sup_{\eta \neq 0} |\eta|^{-\varepsilon} |d^2_\eta a(x,\xi)|$$

$$\Delta^\varepsilon_{x,\xi} a(x,\xi) = \sup_{y,\eta \neq 0} |y|^{-\varepsilon} |\eta|^{-\varepsilon} |d^1_y d^2_\eta a(x,\xi)|.$$

Let m,ρ,δ be real numbers satisfying $0 \leqslant \rho \leqslant 1$, $0 \leqslant \delta < 1$. If k is a non-negative integer and $N = k+\varepsilon$, $0 \leqslant \varepsilon < 1$, we denote by $S^m_{\rho,\delta}(N)$

the space of measurable functions $a(x,\xi)$ defined in $\mathbf{R}^n \times \mathbf{R}^n$ such that its weak derivatives $D_x^\alpha D_\xi^\beta a(x,\xi)$, of order $\alpha,\beta \in \mathbf{N}^n$, $|\alpha| = \alpha_1+\ldots+\alpha_n \leqslant k$, $|\beta| \leqslant k$, are locally integrable functions which satisfy for almost every x, ξ the following estimates

$$(2.1) \qquad |D_x^\alpha D_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

$$(2.2) \qquad \Delta_x^\varepsilon D_x^\alpha D_\xi^\beta a(x,\xi) \leqslant C'_{\alpha,\beta}(1+|\xi|)^{m+\delta(|\alpha|+\varepsilon)-\rho|\beta|}$$

$$(2.3) \qquad \Delta_\xi^\varepsilon D_x^\alpha D_\xi^\beta a(x,\xi) \leqslant C''_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho(|\beta|+\varepsilon)}$$

$$(2.4) \qquad \Delta_{x,\xi}^\varepsilon D_x^\alpha D_\xi^\beta a(x,\xi) \leqslant C'''_{\alpha,\beta}(1+|\xi|)^{m+\delta(|\alpha|+\varepsilon)-\rho(|\beta|+\varepsilon)}$$

Notice that (2.2), (2.3) and (2.4) are superfluous if $\varepsilon=0$. The sum of the best constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$, $C''_{\alpha,\beta}$, $C'''_{\alpha,\beta}$, that appear in (2.1),...,(2.4) is a norm that turns $S^m_{\rho,\delta}(N)$ into a Banach space and will be denoted by $\| \ \|_{S^m_{\rho,\delta}(N)}$.

PROPOSITION 2.1. $S^m_{\rho,\delta}(N)$ *is a Banach space. This space increases if* m *and* δ *increase and* ρ *and* N *decrease. If* $a \in S^m_{\rho,\delta}(N)$ *and* $|\alpha|$, $|\beta| \leqslant N$, *it follows that*

$$D_x^\alpha D_\xi^\beta a \in S^{m-\rho|\beta|+\delta|\alpha|}_{\rho,\delta}(N-\max(|\alpha|,|\beta|))$$

and if $b \in S^{m'}_{\rho,\delta}(N)$, *it follows that* $ab \in S^{m+m'}_{\rho,\delta}(N)$.

We now indicate the proof of $S^m_{\rho,\delta}(N) \subseteq S^m_{\rho,\delta}(N')$ when $N \geqslant N'$. Assume first that $N=\varepsilon$, $N'=\varepsilon'$, $0 \leqslant \varepsilon' < \varepsilon < 1$. Then, if $a \in S^m_{\rho,\delta}(\varepsilon)$ we have, for instance, $|a(x,\xi)| \leqslant C(1+|\xi|)^m$, $|d_y^1 a(x,\xi)| \leqslant \leqslant C|y|^\varepsilon(1+|\xi|)^{m+\delta\varepsilon}$. These estimates imply that

$$|y|^{-\varepsilon'}|d_y^1 a(x,\xi)| \leqslant 2C(1+|\xi|)^m \min(|y|^{-\varepsilon'},(1+|\xi|)^{\delta\varepsilon}|y|^{\varepsilon-\varepsilon'}).$$

The function $f(r) = \min(r^{-\varepsilon'},Ar^{\varepsilon-\varepsilon'})$, $r > 0$, $A > 0$, has a maximum at $r_o = A^{-1/\varepsilon}$ equal to $f(r_o) = A^{\varepsilon'/\varepsilon}$. It follows that $|y|^{-\varepsilon'}|d_y^1 a(x,\xi)| \leqslant 2C(1+|\xi|)^{m+\delta\varepsilon'}$ so $\Delta_x^\varepsilon a(x,\xi) \leqslant 2C(1+|\xi|)^{m+\delta\varepsilon'}$. The other estimates follow in a similar fashion and we obtain $S^m_{\rho,\delta}(\varepsilon) \subseteq S^m_{\rho,\delta}(\varepsilon')$.

If $a \in S^m_{\rho\,\delta}(1)$, the estimate $|D^\beta_\xi a(x,\xi)| \leq C(1+|\xi|)^{m-\rho}$, $|\beta| = 1$, together with the mean value theorem yield

$$(2.5) \qquad |d^2_\eta a(x,\xi)| \leq C(1+|\xi|)^{m-\rho}|\eta| \quad , \quad |\eta| \leq |\xi|+1.$$

On the other hand, from the triangular inequality

$$(2.6) \qquad |d^2_\eta a(x,\xi)| \leq C'(1+|\xi|)^m \quad , \quad \eta \in \mathbf{R}^n.$$

Thus, (2.5) and (2.6) imply

$$\Delta^1_\xi a(x,\xi) \leq C''(1+|\xi|)^{m-\rho} ,$$

as $\rho \leq 1$. Using this estimate and $|a(x,\xi)| \leq C(1+|\xi|)^m$ we get $|d^2_\eta a(x,\xi)| \leq \text{const.}(1+|\xi|)^{m-\rho\varepsilon}|\eta|^\varepsilon$. Similarly, we get the other estimates required to show that $S^m_{\rho,\delta}(1) \subset S^m_{\rho,\delta}(\varepsilon)$. It follows now inductively that

$$S^m_{\rho,\delta}(k+1) \subseteq S^m_{\rho,\delta}(k+\varepsilon) \subseteq S^m_{\rho,\delta}(k+\varepsilon') \subseteq S^m_{\rho,\delta}(k) \quad , \ k \in \mathbf{N} \ , \ 0 < \varepsilon' < \varepsilon < 1.$$

In the next section we will consider the space of Hölder functions $\Lambda_r(\mathbf{R}^n)$. Let us recall some well known facts. If r=0, $\Lambda_o = L^\infty(\mathbf{R}^n)$, if $0 < r < 1$, Λ_r is the subspace of Λ_o of the functions satisfying $|f(x)-f(y)| \leq C|x-y|^r$ a.e., the class of f contains a continuous representative. For general $r > 0$, we write $r = [r]+r-[r] = k+\varepsilon$, $k \in \mathbf{N}$, $0 \leq \varepsilon < 1$, and Λ^r is the space of the functions $f \in \Lambda_o$ with weak derivatives $D^\alpha_x f \in \Lambda^\varepsilon$ for $|\alpha| \leq k$. When $0 < r < 1$, the norm $|||f|||_r$ is the maximum between $\|f\|_\infty$ and the essential supremum of the quotients $|f(x)-f(y)|\,|x-y|^{-r}$. When $r = k+\varepsilon$, $0 \leq \varepsilon < 1$, $k \in \mathbf{N}$, $|||f|||_r = \max_{|\alpha|\leq k} |||D^\alpha f|||_r$.

§ 3. BASIC LEMMAS

The following is a discrete version of a lemma of Alvarez-Calderón ([1], [2]) and may be refered to as the sharp almost-orthogonality principle. We include the proof for completeness.

LEMMA 3.1. *Let* $s > n/2$ *and set* $r = 1-n/2s$. *Then, there is a positive constant* $C = C(s,n)$ *such that for any finite number of functions* $f_k \in H^s$, $k \in \mathbf{Z}^n$, *we have*

(3.1) $$\|\sum_k e_k f_k\|_o^2 < C(\sum_k \|f_k\|_o^2)^r (\sum_k \|f_k\|_s^2)^{1-r} .$$

Here, H^s indicates the Sobolev space in $\mathbf{R}^n$ with norm

$$\|f\|_o^2 = (2\pi)^{-n}\int |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi \quad , \quad \hat{f}(\xi) = \int e^{-ix.s} f(x)\,dx ,$$

and e_k indicates the operator of multiplication by the bounded function $\exp(ik.x)$, $i = \sqrt{-1}$, $x.k = x_1k_1+\ldots+x_nk_n$.

Proof. If

$$\omega_\lambda(\xi) = \sum_{k\varepsilon Z^n} (1+\lambda|\xi-k|^{2s})^{-1} \quad , \quad \text{for} \quad 2s > n ,$$

there is a positive constant $C = C(s,n)$ such that $\omega_\lambda(\xi) \leq C\lambda^{-n/2s}$ for $\xi \in \mathbf{R}^n$ and $0 < \lambda \leq 1$. Then, by Parseval's formula

$$\|\sum_k e_k f_k\|_o^2 \leq (2\pi)^{-n}\int |\sum_k \hat{f}_k(\xi-k)|^2 d\xi \leq$$

$$\leq (2\pi)^{-n}\int \sum_k (1+\lambda(\xi-k)^{2s})|\hat{f}_k(\xi-k)|^2 \omega_\lambda(\xi)\,d\xi \leq$$

$$\leq C\lambda^{-n/2s}(\sum_k \|f_k\|_o^2 + \lambda\sum_k \|f_k\|_s^2) .$$

It is enough to take

$$\lambda = \sum \|f_k\|_o^2 \;/\; \sum \|f_k\|_s^2$$

to obtain (3.1).

Let k be a non-negative integer, ε a real number, $0 < \varepsilon < 1$, and set $s = k+\varepsilon$. It is well known that an equivalent norm for the space H^s is given by

(3.2) $$\|f\|_s^2 = \sum_{|\alpha|\leq k} \|D^\alpha f\|_o^2 + \sum_{|\alpha|=k}\int_{|t|\leq 1} \|D^\alpha(f_t-f)\|_o^2 |t|^{-n-2\varepsilon}dt$$

where $f_t(x) = f(x+t)$.

LEMMA 3.2. *Let* s, N *be real numbers* $n/2 < s < N$ *and consider a symbol* $a(x,\xi) \in S^o_{oo}(N)$, x, $\xi \in \mathbf{R}^n$, *such that* $a(x,\xi) = 0$ *if* $|\xi| \geq \sqrt{n}$. *Then there exists a constant* $C = C(N,s,n)$ *such that*

(3.3) $$\|a(x,D)\|_{\mathcal{L}(L^2)} \leq C \sup_x |||a(x,.)|||_N$$

(3.4) $$\|a(x,D)\|_{\mathcal{L}(L^2,H^s)} \leqslant C \|a\|_{S^o_{oo}(N)} .$$

(The norm $\|\!|\ \|\!|_N$ was defined at the end of §2).

Proof. It is enough to prove the lemma when $s = k+\varepsilon'$, $N = k+\varepsilon$, $0 < \varepsilon < \varepsilon' < 1$, $k \in Z^+$. Setting

$$k(x,y) = (2\pi)^{-n}\int e^{ix.\xi}a(x,\xi)d\xi \quad , \quad \omega(y) = (1+|y|^2)^{-s}$$

we have for $f \in S$

$$|a(x,D)f(x)|^2 = |\int k(x,y)f(x-y)dy|^2 \leqslant \int |k(x,y)|^2\omega^{-1}(y)dy\,.(\omega*|f|^2)(x)$$

$$\leqslant C \|a(x,.)\|_s^2(\omega*|f|^2)(x).$$

Integrating both sides of this estimate we get

(3.5) $$\|a(x,D)\|_{\mathcal{L}(L^2)} \leqslant C \sup_x \|a(x,.)\|_s .$$

Using (3.2) and the fact that $a(x,\xi)$ vanishes for $|\xi| \geqslant \sqrt{n}$, we can estimate $\|a(x,.)\|_s$ by $\|\!|a(x,.)\|\!|_N$. This gives (3.3).
Set $g(x) = a(x,D)f(x)$, $f \in S$. For $|\alpha| \leqslant k$ we may write

(3.6) $$D^\alpha g(x) = a_\alpha(x,D)f(x)$$
$$D^\alpha(g(x+t)-g(x)) = a_\alpha^t(x,D)f(x)$$

with

(3.7) $$a_\alpha(x,\xi) = \sum_{\beta\leq\alpha} \alpha!(\beta!)^{-1}[(\alpha-\beta)!]^{-1}\xi^\beta a(x,\xi)$$
$$a_\alpha^t(x,\xi) = e^{it.\xi}a_\alpha(x+t,\xi)-a_\alpha(x,\xi).$$

Taking account of (3.2), and (3.6), we get

(3.8) $$\|g\|_s^2 \leqslant C(\sum_{|\alpha|\leq k} \|a_\alpha(x,D)\|^2_{\mathcal{L}(L^2)} +$$
$$+ \sum_{|\alpha|=k} \int_{|t|\leq 1} \|a_\alpha^t(x,D)\|^2_{\mathcal{L}(L^2)} |t|^{-n-2\varepsilon}dt)\|f\|_o^2 .$$

Thus, (3.4) follows from (3.8) and the next lemma.

LEMMA 3.3. *With notation* (3.7),

$$\|a_\alpha(x,D)\|_{\mathcal{L}(L^2)} \leq C\,\|a\|_{S^o_{oo}(N)}$$

$$\|a^t_\alpha(x,D)\|_{\mathcal{L}(L^2)} \leq C|t|^{\varepsilon'}\|a\|_{S^o_{oo}(N)}$$

Proof. By (3.3), it is enough to estimate $|\!|\!|a_\alpha(x,.)|\!|\!|_N$ and $|\!|\!|a^t_\alpha(x,.)|\!|\!|_N$. This is easily done using the following

PROPOSITION 3.1. *Let* $f(x,\xi) \in S^o_{oo}(\varepsilon')$, $0 < \varepsilon' < 1$ *and assume that* $f(x,\xi) = 0$ *if* $|\xi| \geq \sqrt{n}$ *and set*

$$f^t(x,\xi) = e^{it.\xi}f(x+t,\xi)-f(x,\xi)$$

Then, there is a positive constant $C = C(n)$ *such that*

$$|f^t(x,\xi)| \leq C(n)(\Delta^{\varepsilon'}_x f(x,\xi)+|f(x,\xi)|)|t|^{\varepsilon'} \tag{3.9}$$

$$|d^2_\eta f^t(x,\xi)| \leq C(n)(\Delta^{\varepsilon'}_{x,\xi}f(x,\xi)+\Delta^{\varepsilon'}_\xi f(x,\xi)+|f(x+t,\xi)|)|t|^{\varepsilon'}|\eta|^{\varepsilon'}\;. \tag{3.10}$$

Proof. We prove (3.10), the proof of (3.9) is simpler. It is easy to check that

$$\begin{aligned} d^2_\eta f^t(x,\xi) &= e^{it.(\xi+\eta)}d^1_t d^2_\eta f(x,\xi)+(e^{it.(\xi+\eta)}-1)d^2_\eta f(x,\xi) + \\ &+ e^{it.\xi}(e^{it.\eta}-1)f(x+t,\xi) \end{aligned} \tag{3.11}$$

(the difference operators d^1_t, d^2_η were defined in § 2). Thus (3.10) follows from the trivial estimate $|e^{i\tau}-1| \leq \min(|\tau|,2)$, $\tau \in \mathbf{R}$.

LEMMA 3.4. *Let* s, N *be real numbers,* $n/2 < s < N$, *and consider a symbol* $a \in S^o_{oo}(N)$ *such that* $a(x,\xi) \equiv 0$ *if* $|\xi|$ *is large enough. Then there exists a positive constant* $C = C(N,s,n)$ *such that*

$$\|a(x,D)\|_{\mathcal{L}(L^2)} \leq C(\sup_x |\!|\!|a(x,.)|\!|\!|_N)^r\|a\|^{1-r}_{S^o_{oo}(N)}\,, \tag{3.12}$$

where $r = 1-n/2s$.

Proof. Set $g = a(x,D)f$, $f \in \mathcal{S}$, and consider a function $\phi \in C^\infty_c(\mathbf{R}^n)$ supported in $|\xi| \leq \sqrt{n}$ such that

$$\sum_{k\in Z^n} \phi^2(\xi-k) = 1.$$

Then g can be written as a finite sum,

$$g = \sum e_k g_k \quad , \quad g_k = a_k(x,D) f_k$$

where

$$\hat{f}_k(\xi) = \phi_k(\xi)\hat{f}(\xi) \quad , \quad a_k(x,\xi) = a(x,\xi+k)\phi(\xi).$$

In particular, it follows from Lemma 3.2, that

$$\| g_k \|_o^2 \leqslant C \sup_x ||| a(x,.) |||_N^2 \| f_k \|^2$$

$$\| g_k \|_s^2 \leqslant C \| a \|^2_{S^o_{oo}(N)} \| f_k \|^2 .$$

Applying Lemma (3.1) to g and observing that $\sum \| f_k \|^2 = \| f \|^2$ (3.12) follows.

§ 4. PROOF OF THEOREM 2

Since $S^m_{\rho,\delta}$ increases with m and δ, we may assume that m=0 and $\delta=\rho$. There is no loss of generality in assuming that $a(x,\xi)$ vanishes if $|\xi| \leqslant 1$ and we do so. Choose a non-negative function $\phi \in C^\infty_c(\mathbf{R}^n)$ supported in $1/3 \leqslant |\xi| \leqslant 1$ and such that $\sum_{j=0}^\infty \phi(2^{-j}\xi) = 1$ if $|\xi| \geqslant 1/2$. The dyadic decomposition of $a(x,\xi)$ is

$$a(x,\xi) = \sum_{j=0}^\infty \phi(2^{-j}\xi) = \sum_{j=0}^\infty a_j(x,\xi). \tag{4.1}$$

Since $|\xi| \sim 2^j$ when (x,ξ) is in the support of a_j we get with $N = k+\varepsilon = [\frac{n}{2}]+\varepsilon$, $0 < \varepsilon < 1$,

$$\begin{aligned} D^\alpha_x D^\beta_\xi a_j(x,\xi) &\leqslant C\ 2^{j\delta(|\alpha|-|\beta|)} \\ \Delta^\varepsilon_x D^\alpha_x D^\beta_\xi a_j(x,\xi) &\leqslant C\ 2^{j\delta(|\alpha|+\varepsilon-|\beta|)} \\ \Delta^\varepsilon_\xi D^\alpha_x D^\beta_\xi a_j(x,\xi) &\leqslant C\ 2^{j\delta(|\alpha|-|\beta|-\varepsilon)} \\ \Delta^\varepsilon_{x,\xi} D^\alpha_x D^\beta_\xi a_j(x,\xi) &\leqslant C\ 2^{j\delta(|\alpha|-|\beta|)} \end{aligned} \tag{4.2}$$

Let $\psi \geqslant 0 \in \mathcal{S}$ be such that $\hat{\psi}(\xi) = 1$ if $|\xi| \leqslant 2^{-4}$ and $\hat{\psi}(\xi) = 0$ if $|\xi| \geqslant 2^{-3}$ and set

$$p_j(x,\xi) = \int a_j(x-y,\xi)\psi(2^j y) 2^{nj} dy$$

$$q_j(x,\xi) = \int (a_j(x-y,\xi)-a_j(x,\xi))\psi(2^j y)2^{nj}dy$$

$$a_j = p_j+q_j .$$

Since $\int\psi = 1$ it is clear that p_j satisfies estimates (4.2) with the same constant C. Therefore, if we set $\tilde{p}_j(x,\xi) = p_j(2^{-\delta j}x,2^{\delta j}\xi)$ we obtain from (4.2)

(4.3) $$\|\tilde{p}_j\|_{S^o_{oo}(N)} \leqslant C$$

(4.4) $$|||\tilde{p}_j(x,.)|||_N \leqslant C$$

with C independent of j. Applying Lemma (3.4) we conclude that $\|\tilde{p}_j(x,D)\|_{\mathcal{L}(L^2)}$ is uniformly bounded in j, and observing that

$$\|\tilde{p}_j(x,D)\|_{\mathcal{L}(L^2)} = \|p_j(x,D)\|_{\mathcal{L}(L^2)},$$

we get $\|p_j(x,D)\|_{\mathcal{L}(L^2)} \leqslant C$. On the other hand, it is easy to chek that if for any $f \in S$, we set $g_j(x) = p_j(x,D)f(x)$ $h_j(x) = p_j^*(x,D)f(x)$, then $\hat{g}_j$ and $\hat{h}_j$ are supported in the annulus $2^{j-2} \leqslant |\xi| \leqslant 2^{j+1}$, where $\hat{u}$ indicates the Fourier transform of u and $p^*(x,D)$ is the adjoint of $p(x,D)$. In particular, $p_j(x,D)p_k^*(x,D) = p_j^*(x,D)p_k(x,D) = 0$ if $|j-k| \geqslant 3$. So we get

(4.5) $$\|\sum_{j=0}^{M} p_j(x,D)\|_{\mathcal{L}(L^2)} \leqslant C \quad , \quad M \in \mathbf{Z}.$$

For the symbols $\tilde{q}_j(x,\xi) = q_j(2^{-\delta j}x,2^{\delta j}\xi)$ we obtain

(4.6) $$\|\tilde{q}_j\|_{S^o_{oo}(N)} \leqslant C$$

(4.7) $$|||\tilde{q}_j(x,.)|||_N \leqslant C\ 2^{(\delta-1)\varepsilon j}$$

Estimate (4.6) is obtained as (4.3). To prove (4.7) observe that for $|\beta| \leqslant k$

$$|D^\beta_\xi q_j(x,\xi)| = |\int d^1_{-y}D^\beta_\xi a_j(x,\xi)\psi(2^j y)2^{nj}dy| \leqslant$$

$$\leqslant C\ 2^{j(\varepsilon-|\beta|)\delta} \int |y|^\varepsilon \psi(2^j y)2^{nj}dy = C\ 2^{j[\varepsilon(\delta-1)-|\beta|\delta]}$$

Analogously,

$$|d_\eta^2 D_\xi^\beta q_j(x,\xi)| = \left|\int d_\eta^2 d_{-y}^1 D_\xi^\beta a(x,\xi)\psi(2^j y)2^{nj}dy\right| \leqslant$$

$$\leqslant C\ 2^{j[\varepsilon(\delta-1)-(|\beta|+\varepsilon)\delta]}|\eta|^\varepsilon \quad , \quad |\beta| \leqslant k.$$

The above estimates imply (4.7). Using (4.6), (4.7) and Lemma 3.4 we obtain

$$\|q_j(x,D)\|_{\mathcal{L}(L^2)} = \|\tilde{q}_j(x,D)\|_{\mathcal{L}(L^2)} \leqslant C\ 2^{j\varepsilon(\delta-1)(1-n/2s)}$$

Thus $\|q_j(x,D)\|_{\mathcal{L}(L^2)}$ is dominated by a geometric convergent series, and together with (4.5), this implies

$$\|\sum_{j=0}^{M} a_j(x,D)\|_{\mathcal{L}(L^2)} \leqslant C.$$

Since $\sum_{j=0}^{M} a_j(x,D)f(x)$ converges to $a(x,D)f$ in S' the proof is complete.

§ 5. NECESSARY CONDITIONS OF REGULARITY

In this section we consider separate regularity in the variables x and ξ. If $N = k+\varepsilon$, $N' = k'+\varepsilon'$, $k,k' \in \mathbf{N}$, $0 \leqslant \varepsilon$, $\varepsilon' < 1$, we define $S^m_{\rho,\delta}(N,N')$ by the following estimates, valid for $|\alpha| \leqslant k$, $|\beta| \leqslant k'$,

$$|D_x^\alpha D_y^\beta a(x,y)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}$$

$$\Delta_x^\varepsilon D_x^\alpha D_\xi^\beta a(x,\xi) \leqslant C'_{\alpha,\beta}(1+|\xi|)^{m+\delta(|\alpha|+\varepsilon)-\rho|\beta|}$$

$$\Delta_\xi^{\varepsilon'} D_x^\alpha D_\xi^\beta a(x,\xi) \leqslant C''_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho(|\beta|+\varepsilon')}$$

$$\Delta_{x,\xi}^{\varepsilon,\varepsilon'} D_x^\alpha D_\xi^\beta a(x,\xi) \leqslant C'''_{\alpha,\beta}(1+|\xi|)^{m+\delta(|\alpha|+\varepsilon)-\rho(|\beta|+\varepsilon')}$$

where we have used the notation of § 2 and $\Delta_{x,\xi}^{\varepsilon,\varepsilon'}a(x,\xi)$ indicates the essential supremum of $|y|^{-\varepsilon}|\eta|^{-\varepsilon'}|d_y^1 d_\eta^2 a(x,\xi)|$, y, $\eta \in \mathbf{R}^n$. We indicate with $S^{-\infty}(N,N')$ the intersection $\bigcap_m S^m_{\rho,\delta}(N,N')$ with the projective limit topology. In the same way we may define $S^m_{\rho,\delta}(\infty,N')$, $S^{-\infty}(\infty,N')$, etc.

The example of Coifman and Meyer ([9]) $a(x,\xi) = (1+|\xi|^2)^{-n/4}e^{ix.\xi-|x|^2}$ in $\mathbf{R}^n \times \mathbf{R}^n$, exhibits a symbol in $S^o_{oo}(\frac{n}{2},\infty)$ for which a(x,D) is unbounded in L^2, showing that lack of regularity in x cannot be compensated for with high regularity in ξ. In this section we prove

THEOREM 3. *Assume that* $a(x,D)$ *is* L^2*-bounded for all* $a(x,\xi)$ *in* $S^{-\infty}(\infty,N)$. *Then* $N \geq \frac{n}{2}$.

Observe that Theorem 2 shows that all symbols in $S^o(N,N')$ yield bounded operators if $N, N' > n/2$. We do not know if all symbols in $S^o(\infty,\frac{n}{2})$ give bounded operators.

Let us denote by $L(N)$ the closed subspace of $S^o_{oo}(\infty,N)$ of those symbols vanishing for $|\xi| \geq \sqrt{n}$. Theorem 3 follows from

LEMMA 5.1. *Assume* $a(x,D)$ *is* L^2*-bounded for all* $a(x,\xi)$ *in* $L(N)$. *Then* $N \geq n/2$.

Proof. We will consider symbols given by sums of exponentials as in [8] and [12]. By the closed graph theorem there is a continuous seminorm p in $S^o_{oo}(\infty,N)$ such that

$$\|a(x,D)\|_{\mathcal{L}} \leq p(a) \quad , \quad a \in L(N) . \tag{5.1}$$

Take $\phi \in C^\infty_c(\mathbf{R}^n)$, equal to one in the cube $\max|\xi_i| \leq 1/4$ and vanishing outside the cube $\max|\xi_i| \leq 1/2$.

For any positive integer λ, set

$$a_\lambda(x,\xi) = \sum_{\alpha \in A_\lambda} e^{-i\lambda^{-1}\alpha . x}\lambda^{-N}\phi(\lambda\xi-\alpha) \tag{5.2}$$

where A_λ is the set of non-negative multi-indices $\alpha \in \mathbf{N}^+$ such that $\max \alpha_i \leq \lambda-1$. In particular, the cardinal of A_λ is λ^n and $a_\lambda(x,\xi)$ vanishes if $\max|\xi_i| \geq 1$. The terms in (5.2) have disjoint supports and it is a simple exercise in Hölder functions to show that if p is a continuous seminor in $S^o_{oo}(\infty,N)$,

$$p(a_\lambda) \leq C \quad , \quad \lambda = 1,2,\ldots \tag{5.3}$$

To estimate the norm of $a_\lambda(x,D)$, take $f_o \in \mathcal{S}$, $\|f_o\|_o = 1$, so that $\hat{f}_o$ is supported in the cube $\max|\xi_i| < 1/4$ and set

$$\hat{f}(\xi) = \sum_{\alpha \in A_\lambda} \hat{f}_o(\lambda\xi-\alpha) .$$

As the terms are orthogonal,

$$\|f\|_o^2 = \sum_{\alpha\varepsilon A_\lambda} \lambda^{-n} \|f_o\|^2 = 1.$$

On the other hand, since $\phi\, \hat{f}_o = \hat{f}_o$,

$$a(x,\xi)\hat{f}(\xi) = \sum_{\alpha\varepsilon A_\lambda} e^{-i\lambda^{-1}\alpha.x}\lambda^{-N}\hat{f}_o(\lambda\xi-\alpha),$$

so

$$g(x) = a(x,D)f(x) = (2\pi)^{-n}\lambda^{n-N}\int e^{ix.\xi}f_o(\lambda\xi)d\xi$$

and

(5.4) $$\|a_\lambda(x,D)\|_{\mathcal{L}(L^2)} \geqslant \|g\|_o = \lambda^{n-2N}.$$

It follows from (5.1), (5.3) and (5.4) that λ^{n-2N} is bounded for $\lambda = 1,2,\ldots$, so $n-2N \leqslant 0$.

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