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ON SOME EXTENSION THEOREMS IN FUNCTIONAL ANALYSIS AND THE THEORY OF BOOLEAN ALGEBRAS

Daniel Gluschankof and Miguel Tilli

ABSTRACT. We present a simplified proof of the equivalence between the Hahn-Banach Theorem and the existence of certain measures on a power-set. Furthermore, by defining a notion of a Boolean integral and applying similar techniques, we prove the corresponding set of equivalences for the Sikorski Extension Theorem (SET).

1. INTRODUCTION

The usual proofs of the Hahn-Banach extension theorem (HB) depend on the Axiom of Choice (AC). Using techniques from non-standard analysis, Luxemburg [8] proved that it can be derived from the Boolean Prime Ideal Theorem (PI), and Pincus showed in [11] that HB is weaker than PI. More precisely, Luxemburg showed, without recourse to AC, the equivalence of HB with the existence of certain measures in power-sets.

In §2 below we present, among other things, a proof of this equivalence using only elementary concepts of functional analysis and measure theory. Our proof yields also the well-known equivalence between HB and Krein's theorem (KT) on the extension of certain positive functionals, as well as the equivalence of HB with an apparently stronger result on the extension of homomorphisms in ordered semigroups due to M.Cotlar [5].

Our method of proof yields a similar chain of equivalences in the theory of Boolean algebras. Namely, we prove the equivalence between:
- an extension theorem (MT) due to Monteiro [10] (which can be seen as a Boolean algebra counterpart of HB);
- the well-known Sikorski extension theorem (SET); cf. [9];
- a "sandwich" extension theorem, due to Cignoli [4].

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(The equivalence between the first two statements was proved by Bacsich [1]).

Now, in order to complete the analogy between the two chains of equivalences, we prove on the functional analysis side a "sandwich" extension theorem for semigroups analogous to Cignoli's and extending Cotlar's result. On the other side of the picture, we introduce a notion of a "Boolean integral" which yields a result similar to Luxemburg's. Of course, all of this is done whithout AC.

It is well known that PI is weaker than AC (see [7]) and it was recently proved by Bell [3] that PI is weaker than SET.
Since it is not known whether SET implies AC, we think it may be useful to have alternative formulations of SET.

2. EXTENSION THEOREMS IN FUNCTIONAL ANALYSIS

We begin by proving (in ZF) the equivalence of the following statements:

HB: Let S be a linear subspace of a real vector space V, p a sublinear functional on V and f a linear functional defined on S such that $f(y) \leqslant p(y)$ $(\forall\, y \in S)$. Then, there exists a linear functional $\tilde{f}$ on V such that $\tilde{f}(x) \leqslant p(x)$ for every x in V and $\tilde{f}(y) = f(y)$ whenever $y \in S$.

KT: Let K be a cone of a vector space V, S a subspace and f a linear functional defined on S such that $f(y) \geqslant 0$ for every y in $K \cap S$. If S contains an internal point of K (that is, a point z such that for all v in $V \setminus K$ the open segment (z,v) intersects K), then there is a linear functional $\tilde{f}$ defined on V satisfying $\tilde{f}(y) = f(y)$ for all y in S and $\tilde{f}(z) \geqslant 0$ for all z in K.

MPS (Measures on power-sets): Let X be a set, $P(X)$ its power-set and I a proper ideal on $P(X)$. Then there exists a finitely additive measure μ on $P(X)$ with values in the interval $[0,1]$ and $\mu(a) = 0$ whenever $a \in I$.

STS (Sandwich theorem on semigroups): Let $(G,+,\leqslant,0)$ be a preordered abelian semigroup, e an element of G, G(e) the subsemigroup defined by $\{g \in G:$ there exist nonnegative integers n,n', positive integers r,r' and z,z',z'',z''' in G such that $ne+z \leqslant \leqslant rg+z'$ and $r'g+z'' \leqslant n'e+z'''\}$.
Let S be a subset of G(e) containing the element e, p and m real-valued, order-preserving, respectively subadditive and superadditive maps on G such that $m(0) = p(0) = 0$, $m(g) \leqslant p(g)$ for all g in G(e) and f a real-valued map defined on S such

that:

For all $(x_1,\dots,x_n) \subset S$, $(y_1,\dots,y_m) \subset S$, z,z' in G,

$\Sigma x_i+z \leqslant \Sigma y_i+z'$ implies $\Sigma f(x_i)+m(z) \leqslant \Sigma f(y_i)+p(z)$ (*)

Then there exists an extension $\tilde{f}$ of f defined on G(e) such that $\tilde{f}$ satisfies (*) for $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ in G(e).

CT (Cotlar Theorem): Let $(G,+,\leqslant,0)$ be a preordered abelian semigroup, e an element of G, G'(e) the subsemigroup defined by $\{g \in G$: there exist nonnegative integers n,n', a positive integer r and $z,z' \in G$ such that $ne \leqslant rg+z$ and $g \leqslant n'e+z'\}$. If S is a subset of G'(e) containing e, p a real-valued, order-preserving subadditive map on G and f a real-valued map defined on S such that:

For all $(x_1,\dots,x_n) \subset S$, $(y_1,\dots,y_m) \subset S$, z in G,

$\Sigma x_i \leqslant \Sigma y_i+z$ implies $\Sigma f(x_i) \leqslant \Sigma f(y_i)+p(z)$ (**)

Then there exists an extension $\tilde{f}$ of f defined on G'(e) such that $\tilde{f}$ satisfies (**) for $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ in G'(e).

REMARKS.

(a) Note that the conditions (*) and (**) of STS and CT trivially imply that f is additive, order-preserving and $m \leqslant f \leqslant p$ when all three are defined. (If G = G(e) then the converse implication holds as well).

(b) The assumptions of both STS and CT may seem somewhat technical; their motivation can be found in Cotlar [5], pp.10-11.

(c) STS and CT stand on a relationship similar to that existing between the statements SET and MT in the theory of Boolean algebras, see Introduction.

PROOFS.

HB ⇒ KT: The standard proof, see e.g. [6], pp.143-146, is done within ZF (i.e. not using AC).

KT ⇒ MPS: Let A be a set. In the linear space R^A we consider the positive cone $K = \{x \in R^A/x_\alpha \geqslant 0$ for all α in A$\}$. We identify $P(A)$ with 2^A, which is contained in R^A, and call 1_Y the characteristic function of $Y \subseteq A$. Let I be a proper ideal of $P(A)$; then 1_A is not in <I> (the subspace of R^A generated by I). For, if $x \in$ <I> then $x = \Sigma c_i x^i$ with $c_i \in R \setminus \{0\}$ and $x^i \in I$ $(i = 1,\dots,n)$. Setting $\mathrm{supp}(x) = \{\alpha \in A: x_\alpha \neq 0$, we have

$\mathrm{supp}(\Sigma c_i x^i) \subseteq \cup\, \mathrm{supp}(x^i) \neq A$ (because I is a proper ideal).
We define a linear functional f on the subspace $S = \langle I\rangle \oplus \langle 1_A\rangle$ by the formulae:

$$f(x) = \begin{cases} 0 & \text{if } x \in \langle I\rangle \\ c & \text{if } x = c.1_A \quad (c \in R) \end{cases}$$

Since 1_A is an internal point of K, KT implies the existence of a linear extension $\tilde{f}$ of f such that $\tilde{f}(z) \geqslant 0$ for all z in K.
Define $\mu: P(A) \to [0,1]$ by $\mu(a) = \tilde{f}(1_a)$. It is easy to see that μ is a measure; clearly, $\mu(a) = 0$ whenever $a \in I$.

MPS $\Rightarrow$ STS: We shall prove this implication in two steps. First, using only ZF, for each point g in $G(e)\setminus S$ we construct an extension of f to $S \cup \{g\}$ satisfying (*). Then, using MPS, we construct an extension defined on the whole of G(e).

Let g be a point in $G(e)\setminus S$; define

$$a = \inf \frac{\Sigma f(y_i) - \Sigma f(x_i) + p(z') - m(z)}{r}$$

where $(x_1,\dots,x_n)$, $(y_1,\dots,y_m) \subset S$, $z,z' \in G$, $r \in N$ and $\Sigma x_i + rg + z \leqslant \Sigma y_i + z'$ holds.
In the same way define

$$b = \sup \frac{\Sigma f(x_i) - \Sigma f(y_i) + m(z) - p(z')}{r}$$

for all $(x_1,\dots,x_n)$, $(y_1,\dots,y_m)$, z,z',r such that $\Sigma x_i + z \leqslant \Sigma y_i + rg + z'$ holds.

We shall show that $a < \infty$: since g is in G(e), there exist $r' \in N$, $n' \in N \cup \{0\}$ and $z'',z''' \in G$, such that $r'g + z'' \leqslant n'e + z'''$; then we have

$$a \leqslant \frac{n'f(e) + p(z''') - m(z'')}{r'} < \infty.$$

In the same way we can see that $b > -\infty$.
Now, let $(x_1,\dots,x_n),(y_1,\dots,y_m),(y'_1,\dots,y'_{m'}),(x'_1,\dots,x'_{n'}) \subset G(e)$, $r,r' \in N$ and $z,z',z'',z''' \in G$ be such that $\Sigma x_i + rg + z \leqslant \Sigma y_i + z'$ and $\Sigma x'_i + z'' \leqslant \Sigma y'_i + r'g + z'''$ holds. We have, then
$r'\,\Sigma x_i + r\Sigma x'_i + r'rg + r'z + rz'' \leqslant r'\,\Sigma y_i + r'z' + r\,\Sigma x'_i + rz'' \leqslant$
$\leqslant r\,\Sigma y'_i + r'\,\Sigma y_i + rr'g + rz''' + r'z'$, and it is easy to prove the inequalities:

$$\frac{\Sigma f(y_i) - \Sigma f(x_i) + p(z') - m(z)}{r} =$$

$$= \frac{r' \Sigma f(y_i) - r' \Sigma f(x_i) + r'p(z') - r'm(z)}{rr'} \geqslant$$

$$\geqslant \frac{\Sigma f(r'y_i) - \Sigma f(r'x_i) + p(r'z') - m(r'z)}{rr'} \geqslant$$

$$\geqslant \frac{\Sigma f(r'y_i) - \Sigma f(r'x_i) + p(r'z'+rz''') - p(rz''') - m(r'z+rz'')+m(rz'')}{rr'} \geqslant$$

$$\geqslant \frac{\Sigma f(rx_i') - \Sigma f(ry_i') - p(rz''') + m(rz'')}{rr'} \geqslant$$

$$\geqslant \frac{\Sigma f(x_i') - \Sigma f(y_i') - p(z''') + m(z'')}{r'} .$$

Then we conclude that $b \leqslant a$. We define the extension $f_g: S \cup \{g\} \to R$ by

$$f_g(z) = \begin{cases} f(z) & \text{if } z \in S \\ \frac{a+b}{2} & \text{if } z = g \end{cases}$$

Looking at the construction it is clear that f_g satisfies (*).

Now, repeating the procedure above, we can construct in ZF an extension $f_x: S \cup x \to R$, for each finite sequence $x = (g_1,\dots,g_n) \subset \subset G(e)$ (ordered in some way).

Let X be the set

$\{x = (g_1,\dots,g_n;r_x): g_1,\dots,g_n \in G(e)$ and r_x is a total order on $x\}$

We define a map $F: G(e) \to R^X$ by posing:

$$F(g)(x) = \begin{cases} f_x(g) & \text{if } g \in S \cup x \\ 0 & \text{if } g \notin S \cup x \end{cases} \qquad (\dagger)$$

For each g in G(e) define $H(g) = \{x \in X: g \notin S \cup x\}$ (‡).

Let I be the ideal on $P(X)$ generated by the family $(H(g))_{g\in G(e)}$.

Assuming that I is not a proper ideal there would be an element of X, $x = (g_1,\dots,g_n;r_x)$, such that $\cup H(g_i) = X$ and hence $x \in H(g_i)$ for some i, $1 \leqslant i \leqslant n$, contradicting the definition of $H(g_i)$. (Here we are only using a finite version of AC!).

Then we can apply MPS and obtain a measure μ such that $\mu(X) = 1$ and $\mu(a) = 0$ whenever $a \in I$.

If g,g' are in $G(e)$, the subset of X where $F(g+g')(x) = F(g)(x)+F(g')(x)$ does not hold is contained in $H(g) \cup H(g') \cup H(g+g')$, whose measure is 0. Then F is additive almost everywhere (for μ).

We also know that, given g in $G(e)$, $m(g) \leqslant F(g)(x) = f_x(g) \leqslant p(g)$ if $g \in S \cup x$. The set

$\{x \in X: F(g)(x) < m(g)$ or $F(g)(x) > p(g)\}$ is contained in $H(g)$.

Then, for each g in $G(e)$, $m(g) \leqslant F(g)(x) \leqslant p(g)$ holds almost everywhere.

In the same way, it is easy to verify that (*) holds for $F(\ )(x)$ almost everywhere.

Since $F(g)$ is a bounded function defined on X, one may construct explicitly (in ZF) a sequence of simple functions $(\tilde{g}_i)_{i\varepsilon N}$ on X such that $\tilde{g}_i \to F(g)$ uniformly (see [6], IV, Lemma 1.4.7, p.247).

Therefore, for each g in $G(e)$, we can define a Riemann-type integral $\int_X F(g)(x)d\mu = \lim_{1\to\infty} \int_X \tilde{g}_i(x)d\mu$. Then, we have the maps

$G(e) \xrightarrow{F} L^1(X,P(X),\mu) \xrightarrow{\int} R$, and the composition $\int \circ F = \int_X F(\ )(x)d\mu$ is an extension of f to the whole of $G(e)$ that satisfies (*):

Let be $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_m) \subset G(e)$, z,z' in G such that $\Sigma x_i+z \leqslant \Sigma y_i+z'$.

We have that $\int_X F(\Sigma x_i)(x)d\mu = \int_{X\setminus Y} F(\Sigma x_i)(x)d\mu + \int_Y F(\Sigma x_i)(x)d\mu =$

$\int_{X\setminus Y} F(\Sigma x_i)(x)d\mu$ where $Y = \bigcup_i H(x_i) \cup \bigcup_i H(y_i) \cup H(\Sigma x_i) \cup H(\Sigma y_i)$.

Similarly, we prove that $\int_X F(\Sigma y_i)(x)\, d\mu = \int_{X\setminus Y} F(\Sigma y_i)(x)d\mu$.

Then, it is easy to verify that

$\Sigma\int F(x_i)(x)d\mu + m(z) \leqslant \Sigma\int F(y_i)(x)d\mu + p(z)$ holds.

STS $\Rightarrow$ HB: We consider the additive (semi-)group underlying the linear space V with the trivial order: $g \leqslant g'$ if and only if $g = g'$. Obviously $V(0)$ coincides with V. Setting $m(g) = -p(-g)$ we are in the conditions of STS. Then we obtain an extension of the linear map f which is a group homomorphism. The subadditive, R-linear map p defines a locally convex topology on V, for which the extension of f is continuous and, therefore, R-linear.

REMARK.

If $G(e)$ is a linear space, the map F defined in (†) is linear; furthermore, the integral is also linear. Then, so is the extension.

This gives a direct proof of MPS $\Rightarrow$ HB without passing through STS.

MPS $\Rightarrow$ CT: In [5] M.Cotlar gives, for each g in $G'(e) \setminus S$, an explicit construction (in ZF) of an extension $f_g\colon S \cup \{g\} \to R$ satisfying (**). Then, repeating the construction of the second part of the proof of MPS $\Rightarrow$ STS, we obtain an extension $\tilde{f}$ of f defined on the whole of $G'(e)$ and satisfying (**).

CT $\Rightarrow$ HB: Same proof as STS $\Rightarrow$ HB.

3. EXTENSION THEOREMS IN THE THEORY OF BOOLEAN ALGEBRAS

PI (Boolean Prime Ideal Theorem): Any proper ideal on a Boolean algebra can be extended to a prime one.

Equivalently, this theorem can be stated:

Let B be a non-trivial Boolean algebra, S a subalgebra and $f\colon S \to 2$ a homomorphism. There exists a homomorphism $\tilde{f}\colon B \to 2$ such that $\tilde{f}(x) = f(x)$ for all x in S.

DEFINITION. Let X be a set, B a Boolean algebra. A *B-valued measure* on $P(X)$ is a Boolean algebra homomorphism $\mu\colon P(X) \to B$.

REMARK.

Let X be a set and B a Boolean algebra. A homomorphism $\phi\colon B^X \to B$ defines a B-valued measure on $P(X)$ by setting $\mu(a) = \phi(1_a)$ where 1_a is the characteristic function of a subset a of X.

DEFINITION. Let B be a Boolean algebra and X,Y sets. A homomorphism $\phi\colon B^X \to B^Y$ is called a *B-homomorphism* iff $\phi(b \wedge h) = b \wedge \phi(h)$ $b \in B$, $h \in B^X$ (identifying $b \in B$ with the constant function b).

LEMMA. Let X be a set, B a complete Boolean algebra, $\phi\colon B^X \to B$ a B-homomorphism, then

$$\phi(h) \geqslant \bigvee_{b \in B} (b \wedge \mu(h^{-1}(b)))$$

for every $h \in B^X$ where μ is the measure induced by ϕ.

Furthermore, if h(X) is finite, then the equality holds.

PROOF. If $h \in B^X$, we can write $h(x) = \bigvee_{b \in B} (b \wedge 1_{h^{-1}(b)}(x))$.

Then $\phi(h) = \phi(\bigvee_{b\in B}(b \wedge 1_{h^{-1}(b)})) \geqslant \bigvee_{b\in B} \phi(b \wedge 1_{h^{-1}(b)}) =$

$$= \bigvee_{b\in B}(b \wedge \mu(h^{-1}(b))).$$

DEFINITION. With the notations of the preceding lemma we shall say that a B-homomorphism from B^X to B is a B-*integral* if $h_{|X\setminus a} = 0$ and $\mu(a) = 0$ imply $\phi(h) = 0$. We denote ϕ by $\int_X d\mu$ and write $\int_X h d\mu$ for $\phi(h)$.

LEMMA. Let $\int_X d\mu$: $B^X \to B$ be a B-integral. If $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$ then $\int_X h d\mu = \int_{X_1}(h_{|X_1})d\mu_1 \vee \int_{X_2}(h_{|X_2})d\mu_2$, where $\int_{X_i} d\mu_i$ $(i = 1,2)$ is the restriction of $\int_X d\mu$ to $\{h \in B^X\colon h_{|X_{3-i}} = 0\}$. (identifying this set with B^X).

PROOF. We set $h_i = h_i \wedge 1_{X_i}$ $(i = 1,2)$. Then $h = h_1 \vee h_2$ implies that $\int_X h d\mu = \int_X h_1 d\mu \vee \int_X h_2 d\mu$, and since $\int_X h_i d\mu$ coincides with $\int_{X_i}(h_{|X_i})d\mu_i$ $(i = 1,2)$, we are done.

Now, using a technique similar to that of §2 we prove the equivalence of the following statements:

SET Let A be a Boolean algebra, B a complete Boolean algebra and f a B-valued homomorphism defined on a subalgebra of A. Then, there exists an extension of f to the whole of A.

BI (Boolean integral) Let X be a set, B a complete Boolean algebra and I a proper ideal on $P(X)$. Then, there exists a B-integral $\int_X d\mu$ defined on B^X such that $\mu_{|I} = 0$.

LET (Lattice extension theorem) Let G be a distributive lattice, B a complete Boolean algebra, S a subset of G containing 0 and 1; j: $G \to B$ and m: $G \to B$ a join - and a meet-homomorphism, respectively, preserving 0 and 1; f: $S \to B$ a homomorphism satisfying $m \leqslant f \leqslant j$ where all three are defined. Then, there exists an extension $\tilde{f}$: $G \to B$ such that $\tilde{f}(g) = f(g)$ for all g in S, and $m \leqslant \tilde{f} \leqslant j$ on G.

MT Let S be a subalgebra of a Boolean algebra G, B a complete Boolean algebra, j: $G \to B$ a join-homomorphism preserving 0 and

1 and f: $S \to B$ a homomorphism satisfying $f \leq j$ on S. Then there exists an extension $\tilde{f}$ of f to the whole of G such that $\tilde{f} \leq j$ on G.

LEMMA. SET implies the conjunction of PI and "every complete Boolean algebra is a retract of its ultrapowers".

PROOF. This can be found in [9] where it is done in ZF.

PROOFS.

SET $\Rightarrow$ BI: Let B be a complete Boolean algebra, X a set and I a proper ideal on $P(X)$. By PI we can extend I to a prime ideal P, whose complement is an ultrafilter U. Since $B^X/_U$ is an ultrapower of B, by the lemma above we have a retract r: $B^X/_U \to B$. We claim that the map $r \circ \pi$: $B^X \to B$, where π is the canonical map π: $B^X \to B^X/_U$, is the required integral. For any $a \subset X$, we have $\mu(a) = r \circ \pi(1_a)$. Since $1_a(x) \in 2$ for all x in X, $\pi(1_a) \in 2 \subseteq B$. Since r is a retract, $\mu(a) = 0$ implies $\pi(1_a) = 0$, and also $\{x \in X / 1_a(x) = 1\} = a \in P$. If, in addition, $h \in B^X$ and $h_{|X \setminus a} = 0$, then $\pi(h) = 0$, and $r \circ \pi(h) = 0$. It is clear that $\mu_{|I} = 0$.

BI $\Rightarrow$ LET: In [4] R.Cignoli proved LET using Zorn's Lemma. However, he proved in ZF that, for each g in $G \setminus S$ there exists an explicit construction for the extension f_g: $S \cup \{g\} \to B$ such that $m(g) \leq f_g(g) \leq j(g)$. Thus it is possible, for each ordered finite subset $x \subseteq G \setminus S$ to construct an extension of f. As in the proof of MPS $\Rightarrow$ STS, we define a map F: $G \to B^X$ (†) and the proper ideal I of $P(X)$ generated by the sets

$$H(g) = \{x \in X: g \notin S \cup x\} \qquad (‡).$$

Now, applying BI, there exists an integral defined on B^X which vanishes on I.

As in the proof of MPS $\Rightarrow$ STS, it is seen that $\int_X F(\ )(x)d\mu$: $G \to B$ is an extension of f meeting the requirements of LET.

The implications LET $\Rightarrow$ MT $\Rightarrow$ SET are well known and trivial.

4. CONCLUDING REMARKS

i) If, in all four statements of §3 we replace the words "complete Boolean algebra" by "the complete Boolean algebra 2", we obtain a new set of equivalent statements. In particular, it is easy to see that SET(2) is equivalent to PI.

ii) The prime ideal theorem for distributive lattices is equivalent to the Boolean prime ideal theorem:

Looking at the specializations of the statements of §3 to the algebra 2, we have that SET(2) implies LET(2), which in turn implies the prime ideal theorem for distributive lattices (set $S=2$, $j(0) = m(0) = 0$, $j(1) = m(1) = 1$, $m(g) = 0$, $j(g) = 1$ if $g \in G\setminus S$, and f the identity of 2).

The interest of this remark lies in the fact that it is not necessary to imbed the distributive lattice into a Boolean algebra in order to prove the implication (see [2]).

iii) The Hahn-Banach theorem can be thought of as a "weak and continuous" form of the Boolean prime ideal theorem: BI(2), which is equivalent to PI, can be stated: "Given a set X and a proper ideal I of $P(X)$, there exists a measure μ on $P(X)$ with values in $\{0,1\}$ and $\mu(a) = 0$ whenever a belongs to I". This obviously implies MPS, which is equivalent to HB.

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Departamento de Matemáticas
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
1428 Buenos Aires

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