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CYCLIC HOMOLOGY OF K[Z/2Z]

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In this note we compute the cyclic homology of K[Z/2Z] where K is a (commutative, with 1) ring in which 2 is not a zero-divisor.

0. CYCLIC HOMOLOGY

0.1. DEFINITION.

We repeat here the definition given in [L-Q].

Let A be a commutative algebra over a commutative ring K, both with unit and let $A^e = A^2 = A \otimes_K A$, the enveloping algebra of A. If we call $A^n = A \otimes_K A \otimes_K \dots \otimes_K A$ (n times) then A^n is a left A^2-module by defining $(a \otimes b)(x_1 \otimes \dots \otimes x_n) = ax_1 \otimes x_t \otimes \dots \otimes x_{n-1} \otimes x_n b$.

We can define now a map $b'\colon A^{n+1} \to A^n$, $n \geq 1$, by $b'(x_0 \otimes \dots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i x_0 \otimes \dots \otimes x_i x_{i+1} \otimes \dots \otimes x_n$, which actually is an A^2-module homomorphism.

It is easy to check that $(b')^2 = 0$, so we have a complex $H'(A)$, where $H'_n(A) = A^{n+1}$, $n \geq 0$, with b' as differential.

The map $\varepsilon_0\colon A^n \to A^{n+1}$ $(n \geq 0)$ defined by $\varepsilon_0(x) = 1 \otimes x$, which is just a K-map, satisfies $\varepsilon_0 b' + b'\varepsilon_0 = \mathrm{Id}_{(A^n)}$, hence it is a homotopy so H' is acyclic.

Let $H(A) = A \otimes_{A^e} H'(A)$, i.e., $H_n(A) = A \otimes_{A^2} A^{n+2}$ $(n \geq 0)$ with a differential $b = \mathrm{Id} \otimes b'$. Since A^2 operates at both ends of A^{n+2} we can

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identify $A \otimes_{A^2} A^{n+2}$ with A^{n+1} by the map γ:

$$\gamma(a \otimes_{A^2} b_1 \otimes_K \dots \otimes_K b_{n+2}) = b_{n+2} a b_1 \otimes \dots \otimes b_{n+1}$$

hence

$$b(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i \, a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n \, a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$$

So $H_n(A) = A^{n+1}$ and the differential in $H(A)$ is b. The homology of $H(A)$ is called the Hochschild homology of A.

REMARK 1.

If A is a projective K-module then $H'(A)$ is an A^e-projective resolution of A so $H_n(A) = H_n(H(A)) = \mathrm{Tor}_n^{A^e}(A,A)$.

Let us consider now the K-maps $t, N: A^n \to A^n$ defined by

$$t(a_1 \otimes \dots \otimes a_n) = (-1)^{n-1} \, a_n \otimes a_1 \otimes \dots \otimes a_{n-1} \quad \text{and} \quad N(x) = \sum_{i=0}^{n-1} t^i(x).$$

Direct computations show that $1-t: H'(A) \to H(A)$ and $N: H(A) \to H'(A)$ are morphisms of complexes, i.e. $b'N = Nb$ and $b(1-t) = (1-t)b'$ and $(1-t)N = N(1-t) = 0$, hence we can define a double complex $C(A)$ by

$$C(A)_{p,q} = \begin{cases} 0 & \text{if } p \text{ or } g < 0. \\ A^{q+1} & \text{if } p \geqslant 0, q \geqslant 0. \end{cases}$$

and maps $v: C(A)_{p,q+1} \to C(A)_{p,q}$ $\begin{cases} = b & \text{if } p \text{ is even.} \\ = -b' & \text{if } p \text{ is odd.} \end{cases}$

$h: C(A)_{p+1,q} \to C(A)_{p,q}$ $\begin{cases} = 1-t & \text{if } p \text{ is even.} \\ = N & \text{if } p \text{ is odd.} \end{cases}$

and obtain the following picture

$$
\begin{tikzcd}
0 & A^{p+1} \arrow[l] \arrow[d, "b"'] & A^{p+1} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{p+1} \arrow[l, "N"'] \arrow[d, "b"'] & A^{p+1} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\
0 & A^{p} \arrow[l] \arrow[d] & A^{p} \arrow[l, "1-t"] \arrow[d] & A^{p} \arrow[l, "N"] \arrow[d] & A^{p} \arrow[l, "1-t"] \arrow[d] & \cdots \arrow[l] \\
 & {} & {} & {} & {} & \\
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 & A^{2} \arrow[l] \arrow[d, "b"'] & A^{2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \cdots \arrow[l, "N"'] \\
0 & A \arrow[l] & A \arrow[l, "1-t"] & A \arrow[l, "N"] & A \arrow[l, "1-t"] & \cdots \arrow[l, "N"]
\end{tikzcd}
$$

If we call Tot $C(A)$ the total complex associated to the double complex $C(A)$ then we shall define the cyclic homology $HC_n(A)$ by

$$HC_n(A) = H_n(\text{Tot } C(A)) \qquad n \geqslant 0.$$

0.2. RELATION WITH HOCHSCHILD HOMOLOGY

In this section we want to give a brief proof of the following well known result:

THEOREM 0.1. *There is an exact sequence*

$$\cdots \xrightarrow{B} H_n(A) \xrightarrow{i} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} H_{n-1}(A) \xrightarrow{i} HC_{n-1}(A) \xrightarrow{S} \cdots$$

called the SBi *sequence.*

Proof. We can replace the Hochschild complex $H(A)$ by $L(A)$ defined by

$$L_n(a) = A^{n+1} \oplus A^n \qquad L_n(A) \xrightarrow{\partial} L_{n-1}(A) \qquad \partial(x,y) = (bx+N(y),-b'(y)).$$

It is easy to check that $H(A)$ and $L(A)$ are quasi-isomorphic.

On the other hand, $L(A)$ is a subcomplex of the double complex $C(A)$ and its cokernel is again isomorphic to $C(A)$ but the canonical projection is homogeneous of degree-2, hence the exact sequence follows from

$$0 \to (A) \xrightarrow{i} \text{Tot } C(A) \xrightarrow{p} \text{Tot } C(A)[-2] \to 0.$$

The study of this exact sequence will be used to compute the group $HC_n(K(G))$ for $G = \mathbf{Z}/2\mathbf{Z}$.

1. HOCHSCHILD HOMOLOGY FOR SOME GROUP RINGS.

1.1. GENERAL COMMENTS.

In the case of the group ring $A = K(G)$ of a cyclic group $G = \mathbf{Z}/n\mathbf{Z}$, and assuming n is not a zero divisor in K, the Hochschild homology is easily computable by using a simpler A^e-projective resolution as follows:

$$(1) \qquad \cdots \to A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{u} A \to 0$$

where u is the multiplication map, i.e. $u(x \otimes y) = xy$, d_1, d_2 are defined by $d_1(x \otimes y) = (x \otimes y)(g \otimes 1 - 1 \otimes g)$ for g a generator of $\mathbf{Z}/n\mathbf{Z}$, $d_2(x \otimes y) = (x \otimes y)(\Sigma g^i \otimes g^{n-1})$. The sequence is exact (C.E.Chapter XII) hence it is a projective resolution of A. To compute the Hochschild homology (in this case $= \mathrm{Tor}^{A^e}(A,A)$) we just tensorize $\otimes_{A^e} A$ and the sequence becomes

$$(2) \qquad \cdots \to A \xrightarrow{n} A \xrightarrow{0} A \xrightarrow{n} A \xrightarrow{0} A$$

hence $H_0(A) = A$, $H_{2n-1}(A) = A/nA$ and $H_{2n}(A) = 0$ (n is not a zero divisor in $A = K[G]$ because it is not in K).

1.2. COMPARISON BETWEEN THE PROJECTIVE RESOLUTIONS.

Using the projective resolution (1) and so the complex (2) it is easy to see when a cycle is a boundary. The canonical Hochschild complex H' is also, in this case, a projective resolution of A as an A^e-module, hence there is a map $f: H' \to (1)$ which induces $\overline{f}: H \to (2)$ and this map gives an isomorphism in homology which is independent of f.

In the case $G = \mathbf{Z}/2\mathbf{Z} = \{1,g\}$ with $g^2 = 1$ it is easier to define d_2 as $d_2(x \otimes y) = (x \otimes y)(g \otimes 1 + 1 \otimes g)$.

The map f will be defined as an A^2 map such that

$f(1 \otimes g^{\alpha_1} \otimes \cdots \otimes g^{\alpha_r} \otimes 1) = (\alpha_1 \cdots \alpha_r) \otimes 1$ (the product in $\mathbf{Z}$) where $\alpha_i = 0,1$, hence it will be 0 if some $g^{\alpha_1} = 1$ or 1 if all $g^{\alpha_i} = g$.

So $\bar{f}(1\otimes g^{\alpha_1}\otimes,\dots,\otimes g^{\alpha_r}) = (\alpha_1\dots\alpha_r)$ and then we can check, in the sequence H, when a cycle is not a boundary, i.e. when the coefficient of $1\otimes g\otimes,\dots,\otimes g$ is not in $2K$.

NOTATION

If $\alpha\in A^r$ $\alpha = \Sigma\, a_{i_1,\dots,i_r}\ g^{i_1}\otimes\dots\otimes g^{i_r}$ $i_j = 0,1$ then we shall denote by $g\alpha\otimes g$ (resp. $\alpha\otimes 1$) the element of A^{r+1} of the form $g^{i_1+1}\otimes\dots\otimes g^{i_r}\otimes g$ (resp. $(g^{i_1}\otimes\dots\otimes g^{i_r}\otimes g^0)$.

LEMMA 1.1. *Let* $\alpha\in A^r$ *be a* b-*cycle, then* $2\alpha = b[(-1)^{r+1}[g\alpha\otimes g+\alpha\otimes 1]]$.

In fact, if $b\alpha = 0$, $b\alpha = b'\alpha + (-1)^{r+1}\mu_0 t\alpha = 0$ where $\mu_0(a_1\otimes\dots\otimes a_r) = a_1a_2\otimes\dots\otimes a_r$. So, if we call $\gamma = g\alpha\otimes g+\alpha\otimes 1$ we have $b\gamma = gb'\alpha\otimes g+(-1)^r g\alpha g + (-1)^{r+1}\alpha + b'\alpha\otimes 1 + (-1)^r\alpha + (-1)^{r+1}\alpha =$ $= (-1)^{r+1}g\mu_0 t\alpha\otimes g + (-1)^r g\alpha g + (-1)^{r+1}\alpha + (-1)^{r+1}\mu_0 t\alpha\otimes 1 +$ $+ (-1)^r\alpha + (-1)^{r+1}\alpha$ but $g\mu_0 t\alpha\otimes g + \mu_0 t\alpha\otimes 1 = g\alpha g + \alpha$ (in fact if $\alpha = a\, g^{i_0}\otimes\dots\otimes g^{i_n}$ both sums are the sum of all terms $af_0\otimes g^{i_1}\otimes\dots\otimes g^{i_{n-1}}\otimes f_n$ with $f_0f_n = g^i$, so $b((-1)^{r+1})\gamma = 2\alpha)$.

2. CYCLIC HOMOLOGY FOR K[Z/2Z]

2.1. THE DECOMPOSITION OF THE DOUBLE COMPLEX.

In the double complex used to compute the cyclic homology each A^n is the direct sum $A^n = (A^n)^1\oplus(A^n)^g$ where $(A^n)^1$ (resp$(A^n)^g$) is the sub-K-module of A^n generated by the elements $g^{\alpha 1}\otimes\dots\otimes g^{\alpha n}$ with $\Sigma\,\alpha_i = 0$ (resp. $\Sigma\,\alpha_i = 1$) (mod.2).

Since the maps b', b, 1-t and N respect this decomposition we obtain two double complexes $C(K[Z/2Z])^1$ and $C(K[Z/2Z])^g$ and also two Hochschild complexes $[H(K(Z/2Z))]^1$ and $[H(K(Z/2Z))]^g$.

Then both, Hochschild and cyclic homologies, are decomposed as the direct sum of the homologies of the corresponding complexes.

Then, the Hochschild homology $H_n^{(g)}(K(Z/2Z)) =$

$$= H_n^{(1)}(K(\mathbf{Z}/2\mathbf{Z})) = \begin{cases} K & \text{if } n = 0 \\ K/2K & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

On the other hand, Karoubi [K] computed the homology of $C(K(\mathbf{Z}/2\mathbf{Z}))^1$ which is $HC_n^{(1)}(K(\mathbf{Z}/2\mathbf{Z})) = H_n(\mathbf{Z}/2\mathbf{Z}) \oplus H_{n-2}(\mathbf{Z}/2\mathbf{Z}) \oplus \ldots$ and our method consists in providing that $HC_n^{(g)}(K(\mathbf{Z}/2\mathbf{Z})) = \begin{cases} K & \text{if } n=0 \\ 0 & \text{if } n\neq 0. \end{cases}$

We shall call B the composite $(1-t)\,\varepsilon_0 N$.

LEMMA 2.1. *Let* $(\alpha_r,\alpha_{r-1},\ldots,\alpha_0)(\alpha_i \in A^i)$ *be a cycle in* $\mathrm{Tot}C(A)$ *representing an element* β *in* $HC_{r-1}(A)$; *then the image of* β *in* $H_{r+1}(A)$ *is* $B\alpha_r$.

Proof. As usual, we complete a cycle to a chain in $\mathrm{Tot}_{r+2}C(A)$, for instance $\beta = (0,0,\alpha_r,\alpha_{r-1},\ldots,\alpha_0)$, and compute its boundary $(0,N\alpha_r,0,\ldots,0)$ (because $b\alpha_r = (1-t)\alpha_{r-1},\ldots)$ hence $b'N\alpha_r = Nb\alpha_r = N(1-t)\,\alpha_{r-1} = 0$, so $N\alpha_r = b'\varepsilon_0(N\alpha_r)$ and $(0,N\alpha_r) \sim (B\alpha_r,0)$ in $L(A)$.

PROPOSITION 2.2. *Assume* $HC^g_{2n-2}(A)$ *is a free cyclic* K*-module generated by an element* β *which is represented by a cocycle* $(\alpha_{2n-1},\alpha_{2n-2},\ldots,\alpha_0)$ *such that* $\alpha_{2n-1} \in A^{2n-1}$ *has integral odd coefficient in* $g \otimes \ldots\ldots \otimes g$, *then* $B(\alpha_{2n-1})$ *is the generator of* $H^g_{2n-1}(A)$.

Proof. If the coefficient of $g \otimes \ldots \otimes g$ is in $\mathbf{Z}$ and odd, by applying N we multiply it by 2n-1, which is again integral and odd; then we have $1 \otimes g \otimes \ldots\ldots \otimes g$ by ε_0 and by acting with t-1 we keep the same coefficients. All other terms contain a 1 so, by applying ε_0, that 1 goes "inside" and the same happens after (1-t). Hence $f(B(\alpha_{n-1})) = f(1 \otimes \ldots\ldots \otimes g) = g$ which is the generator of $H^g_{2n-1}(A)$.

PROPOSITION 2.3. *Under the same hypothesis of Proposition 1,* $H^g_{2n}(A)$ *is again a free cyclic* K*-module generated by an element* $\overline{\beta}$ *which is represented by a cocycle* $(\overline{\alpha}_{2n+1},\overline{\alpha}_{2n},\ldots,\overline{\alpha}_0)$ *such that* $\overline{\alpha}_i \in A^i$ *and* $\overline{\alpha}_{2n+1}$ *has integral odd coefficient in* $g \otimes \ldots \otimes g$.

Proof. Since $B\colon HC^g_{2n-2}(A) \to H^g_{2n-1}(A)$ is surjective and HC^g_{2n-2} is cyclic generated by an element γ, Ker B is again cyclic generated by 2γ. Since 2 is not a zero divisor in K the ideal (2) in K is again free.

Now $B(2\alpha_{2n-1}) = 2h\ 1 \otimes \dots \otimes g$ + elements in Ker f and $h \in \mathbf{Z}$ (h odd). But $bB(\alpha_{2n-1}) = 0$ because $bB(\alpha_{2n-1}) = b(1-t)\,\varepsilon_0 N(\alpha_{2n-1}) = (1-t)b'\varepsilon_0 N(\alpha_{2n-1}) = (1-t)N(\alpha_{2n-1}) = 0$ hence $2B(\alpha_{2n-1})$ is a b-boundary, i.e. $2B(\alpha_{2n-1}) = b(hg \otimes \dots \otimes g + 1 \otimes g \dots \otimes g \otimes 1 + \text{terms in ker f})$. (Lemma 1.1).

THEOREM 2.4. *If* 2 *is not a zero divisor in* K *the cyclic homology of* $A = K[\mathbf{Z}/2\mathbf{Z}]$ *is*

$$HC_{2n}(A) \cong K \oplus K$$

$$HC_{2n-1}(A) \cong K/2K \oplus \dots \oplus K/2K \quad (n\ \textit{times}).$$

Proof. According to Karoubi's result [K] and the previous comments it will be enough to prove: $HC^g_{2n}(A) \cong K$, $HC^g_{2n-1}(A) = 0$.

The value $HC^g_0(A) = K$ follows by direct computations. According to proposition 2.3. $HC^g_{2n}(A) = 2.\ HC^g_{2n-2}(A)$ and from $HC^g_{2n-2}(A) \cong K$ follows $HC^g_{2n}(A) \cong K$ (since 2 is not a zero divisor in K).

For the odd dimensional homologies we use the exactness of

$$HC^g_{2n-2}(A) \to H^g_{2n-1}(A) \to HC^g_{2n-1}(A) \to HC^g_{2n-3}(A) \to 0$$

and the fact that the first arrow is surjective (Prop.2.1). Hence $H^g_{2n-1}(A) \cong H^g_{2n-3}(A)$ and for n=1, $H^g_{-1}(A) = 0$ so the result follows by induction.

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