Cyclic Homology of $K[Z/2Z]$

Guillermo H. Cortiñas [*] and Orlando E. Villamayor

In this note we compute the cyclic homology of $K[Z/2Z]$ where $K$ is a (commutative, with 1) ring in which $Z$ is not a zero-divisor.

0. CYCLIC HOMOLOGY

0.1. DEFINITION.

We repeat here the definition given in [L-Q].

Let $A$ be a commutative algebra over a commutative ring $K$, both with unit and let $A^e = A^2 = A \otimes_K A$, the enveloping algebra of $A$. If we call $A^n = A \otimes_K A \otimes_K \ldots \otimes_K A$ (n times) then $A^n$ is a left $A^2$-module by defining $(a \otimes b)(x_1 \otimes \ldots \otimes x_n) = ax_1 \otimes x_2 \otimes \ldots \otimes x_{n-1} \otimes bx_n$.

We can define now a map $b': A^{n+1} \to A^n$, $n \geq 1$, by $b'(x_0 \otimes \ldots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i x_0 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x_n$, which actually is an $A^2$-module homomorphism.

It is easy to check that $(b')^2 = 0$, so we have a complex $H'(A)$, where $H'_n(A) = A^{n+1}$, $n \geq 0$, with $b'$ as differential.

The map $\varepsilon_0: A^n \to A^{n+1}$ ($n \geq 0$) defined by $\varepsilon_0(x) = 1 \otimes x$, which is just a $K$-map, satisfies $\varepsilon_0 b' + b' \varepsilon_0 = \text{Id}_{A^n}$, hence it is a homotopy so $H'$ is acyclic.

Let $H(A) = A \otimes_{A^e} H'(A)$, i.e., $H_n(A) = A \otimes_{A^e} A^{n+2}$ ($n \geq 0$) with a differential $b = \text{Id} \otimes b'$. Since $A^2$ operates at both ends of $A^{n+2}$ we can

[*] Supported by a fellowship of the Comisión de Investigaciones Científicas de la Provincia de Buenos Aires.
identify $A \odot_2 A^{n+2}$ with $A^{n+1}$ by the map $\gamma$:

$$\gamma(a \odot_2 b_1 \odot \ldots \odot_k b_{n+2}) = b_{n+2}ab_1 \odot \ldots \odot b_{n+1}$$

hence

$$b(a_0 \odot \ldots \odot a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \odot \ldots \odot a_i a_{i+1} \odot \ldots \odot a_n +
+ (-1)^n a_{n} a_0 \odot a_1 \odot \ldots \odot a_{n-1}$$

So $H_n(A) = A^{n+1}$ and the differential in $H(A)$ is $b$. The homology of $H(A)$ is called the Hochschild homology of $A$.

**Remark 1.**

If $A$ is a projective $K$-module then $H'(A)$ is an $A^e$-projective resolution of $A$ so $H_n(A) = H_n(H(A)) = \text{Tor}_{n} (A^e, A)$.

Let us consider now the $K$-maps $t, N: A^n \to A^n$ defined by

$$t(a_1 \odot \ldots \odot a_n) = (-1)^{n-1} a_n \odot a_1 \odot \ldots \odot a_{n-1} \quad \text{and} \quad N(x) = \sum_{i=0}^{n-1} t^i(x).$$

Direct computations show that $1-t: H'(A) \to H(A)$ and $N: H(A) \to H'(A)$ are morphisms of complexes, i.e. $b'N = Nb$ and $b(1-t) = (1-t)b'$ and $(1-t)N = N(1-t) = 0$, hence we can define a double complex $C(A)$ by

$$C(A)_{p,q} = \begin{cases} 
0 & \text{if } p \text{ or } q < 0, \\
A^{q+1} & \text{if } p \geq 0, q \geq 0.
\end{cases}$$

and maps $v: C(A)_{p,q+1} \to C(A)_{p,q}$

$$v = \begin{cases} 
b & \text{if } p \text{ is even}, \\
-b' & \text{if } p \text{ is odd}.
\end{cases}$$

$$h: C(A)_{p+1,q} \to C(A)_{p,q}$$

$$h = \begin{cases} 
1-t & \text{if } p \text{ is even}, \\
N & \text{if } p \text{ is odd}.
\end{cases}$$

and obtain the following picture.
If we call $\text{Tot} \ C(A)$ the total complex associated to the double complex $C(A)$ then we shall define the cyclic homology $HC_n(A)$ by

$$HC_n(A) = H_n(\text{Tot} \ C(A)) \quad n \geq 0.$$ 

0.2. RELATION WITH HOCHSCHILD HOMOLOGY

In this section we want to give a brief proof of the following well known result:

THEOREM 0.1. There is an exact sequence

$$\ldots \rightarrow B_n(A) \stackrel{\partial}{\rightarrow} H_n(A) \stackrel{i}{\rightarrow} HC_n(A) \stackrel{S}{\rightarrow} HC_{n-2}(A) \rightarrow H_{n-1}(A) \stackrel{i}{\rightarrow} HC_{n-1}(A) \rightarrow \ldots$$

called the $SB_i$ sequence.

Proof. We can replace the Hochschild complex $H(A)$ by $L(A)$ defined by

$$L_n(a) = A^{n+1} \otimes A^n \quad L_n(A) \xrightarrow{\partial} L_{n-1}(A) \quad \partial(x,y) = (bx+N(y), -b'(y)).$$

It is easy to check that $H(A)$ and $L(A)$ are quasi-isomorphic.

On the other hand, $L(A)$ is a subcomplex of the double complex $C(A)$ and its cokernel is again isomorphic to $C(A)$ but the canonical projection is homogeneous of degree-2, hence the exact sequence follows from

$$0 \rightarrow (A) \xrightarrow{i} \text{Tot} \ C(A) \xrightarrow{p} \text{Tot} \ C(A)[-2] \rightarrow 0.$$
The study of this exact sequence will be used to compute the group $HC_n(K(G))$ for $G = Z/2Z$.

1. HOCHSCHILD HOMOLOGY FOR SOME GROUP RINGS.

1.1. GENERAL COMMENTS.

In the case of the group ring $A = K(G)$ of a cyclic group $G = Z/nZ$, and assuming $n$ is not a zero divisor in $K$, the Hochschild homology is easily computable by using a simpler $A^e$-projective resolution as follows:

\[
\begin{align*}
\ldots & \to A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{u} A \to 0
\end{align*}
\]

where $u$ is the multiplication map, i.e., $u(x \cdot y) = xy$, $d_1, d_2$ are defined by $d_1(x \cdot y) = (x \cdot y)(g \cdot 1 - 1 \cdot g)$ for $g$ a generator of $Z/nZ$, $d_2(x \cdot y) = (x \cdot y)(\Sigma g^i \cdot g^{n-1})$. The sequence is exact (C.E. Chapter XII) hence it is a projective resolution of $A$. To compute the Hochschild homology (in this case $\text{Tor}^A_e(A, A)$) we just tensorize $A$ and the sequence becomes

\[
\begin{align*}
\ldots & \to A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{u} A \to 0
\end{align*}
\]

hence $H_0(A) = A$, $H_{2n-1}(A) = A/nA$ and $H_{2n}(A) = 0$ ($n$ is not a zero divisor in $A = K[G]$ because it is not in $K$).

1.2. COMPARISON BETWEEN THE PROJECTIVE RESOLUTIONS.

Using the projective resolution (1) and so the complex (2) it is easy to see when a cycle is a boundary. The canonical Hochschild complex $H'$ is also, in this case, a projective resolution of $A$ as an $A^e$-module, hence there is a map $f: H' \to (1)$ which induces $\bar{f}: H \to (2)$ and this map gives an isomorphism in homology which is independent of $f$.

In the case $G = Z/2Z = \{1, g\}$ with $g^2 = 1$ it is easier to define $d_2$ as $d_2(x \cdot y) = (x \cdot y)(g \cdot 1 + 1 \cdot g)$.

The map $f$ will be defined as an $A^e$ map such that

$f(1 \cdot g^{a_1} \cdot \ldots \cdot g^{a_r} \cdot 1) = (a_1 \ldots a_r) \cdot 1$ (the product in $Z$) where $a_i = 0, 1$, hence it will be 0 if some $g^{a_i} = 1$ or 1 if all $g^{a_i} = g$. 

So \( f(1 \otimes g^{a_1} \otimes \ldots \otimes g^{a_r}) = (a_1 \ldots a_r) \) and then we can check, in the sequence \( \mathcal{H} \), when a cycle is not a boundary, i.e. when the coefficient of \( 1 \otimes g \otimes \ldots \otimes g \) is not in \( 2K \).

**NOTATION**

If \( \alpha \in A^r \), \( \alpha = \sum a_{i_1} \otimes \ldots \otimes a_{i_r} \) and \( i_j = 0,1 \) then we shall denote by \( g_{a_1} \otimes g \cdots \otimes g_{a_r} \) (resp. \( a \otimes 1 \)) the element of \( A^{r+1} \) of the form \( i_1^{a_1} \otimes \ldots \otimes i_r^{a_r} \) (resp. \( (i_1 \otimes \ldots \otimes i_r)^{a} \otimes g^0 \)).

**LEMMA 1.1.** Let \( \alpha \in A^r \) be a b-cycle, then \( 2\alpha = b[(-1)^{r+1}(g_{a_1} \otimes g + a \otimes 1)] \).

In fact, if \( b\alpha = 0 \), \( b\alpha = b'\alpha + (-1)^{r+1} \mu_0 \alpha = 0 \) where

\[
\mu_0(a_1 \otimes \ldots \otimes a_r) = a_1a_2 \otimes \ldots \otimes a_r.
\]

So, if we call \( \gamma = g_{a_1} \otimes g + a \otimes 1 \), we have by \( \gamma = gb'\alpha \otimes g + (-1)^{r+1} \alpha + b' \alpha \otimes 1 \) - \( (-1)^{r+1} \alpha = (-1)^{r+1} g_{\nu_0} \alpha \otimes g + (-1)^{r+1} \alpha + \mu_0 \alpha \otimes 1 + (-1)^{r+1} \). But \( g_{\nu_0} \alpha \otimes g + \mu_0 \alpha \otimes 1 = g \alpha + \alpha \) (in fact if \( \alpha = a g \otimes \ldots \otimes g \) both sums are the sum of all terms \( af_0 \otimes g \otimes \ldots \otimes g f_{n-1} \otimes f_n \) with \( f_0 f_n = g^1 \), so \( b[(-1)^{r+1}] = 2\alpha \).

2. CYCLIC HOMOLOGY FOR \( K[\mathbb{Z}/2\mathbb{Z}] \)

2.1. THE DECOMPOSITION OF THE DOUBLE COMPLEX.

In the double complex used to compute the cyclic homology each \( A^n \) is the direct sum \( A^n = (A^n)^{1} \otimes (A^n)^{2} \) where \( (A^n)^{1} \) (resp. \( (A^n)^{2} \)) is the sub-K-module of \( A^n \) generated by the elements \( g_{a_1} \otimes \ldots \otimes g_{a_n} \) with \( \sum a_i = 0 \) (resp. \( \sum a_i = 1 \) (mod.2)).

Since the maps \( b' \), \( b \), \( 1-t \) and \( N \) respect this decomposition we obtain two double complexes \( C(K[\mathbb{Z}/2\mathbb{Z}])^1 \) and \( C(K[\mathbb{Z}/2\mathbb{Z}])^2 \) and also two Hochschild complexes \( [H(K[\mathbb{Z}/2\mathbb{Z}])]^1 \) and \( [H(K[\mathbb{Z}/2\mathbb{Z}])]^2 \).

Then both, Hochschild and cyclic homologies, are decomposed as the direct sum of the homologies of the corresponding complexes.

Then, the Hochschild homology \( H_n^{(g)}(K[\mathbb{Z}/2\mathbb{Z}]) = \)
\[ H_n^{(1)}(K(Z/2Z)) = \begin{cases} K & \text{if } n = 0 \\ K/2K & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \]

On the other hand, Karoubi [K] computed the homology of \( C(K(Z/2Z)) \)
which is \( H_n^{(1)}(K(Z/2Z)) = H_n(Z/2Z) \oplus H_{n-2}(Z/2Z) \oplus \ldots \)
and our method consists in providing that \( H_n^{(g)}(K(Z/2Z)) = \begin{cases} K & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases} \)

We shall call \( B \) the composite \((1-t) \epsilon_0 N.\)

**Lemma 2.1.** Let \((a_r, a_{r-1}, \ldots, a_0) (a_i \in A^i)\) be a cycle in \( \text{Tot}C(A) \)
representing an element \( \beta \) in \( H_{r+1}(A) \); then the image of \( \beta \) in
\( H_{r+2}(A) \) is \( Ba_r. \)

**Proof.** As usual, we complete a cycle to a chain in \( \text{Tot}C(A) \), for
instance \( \beta = (0, 0, a_r, a_{r-1}, \ldots, a_0) \), and compute its boundary
\( (0, Na_r, 0, \ldots, 0) \) (because \( Ba_r = (1-t)a_{r-1}, \ldots \) hence \( b'Na_r = Nba_r = N(1-t)a_{r-1} = 0 \), so \( Na_r = b'\epsilon_0(Na_r) \) and \( (0, Na_r) \sim (Ba_r, 0) \) in
\( L(A). \)

**Proposition 2.2.** Assume \( H_{2n-2}(g) \) is a free cyclic \( K \)-module generated by an element \( \beta \) which is represented by a cocycle
\( (a_{2n-1}, a_{2n-2}, \ldots, a_0) \) such that \( a_{2n-1} \in A^{2n-1} \) has integral odd coefficient in \( g \cdot \ldots \cdot g \), then \( B(a_{2n-1}) \) is the generator of \( H_{2n-1}(g) \).

**Proof.** If the coefficient of \( g \cdot \ldots \cdot g \) is in \( Z \) and odd, by
applying \( N \) we multiply it by \( 2n-1 \), which is again integral and odd;
then we have \( 1 \cdot g \cdot \ldots \cdot g \) by \( \epsilon_0 \) and by acting with \( t-1 \) we keep
the same coefficients. All other terms contain a 1 so, by applying \( \epsilon_0 \), that 1 goes "inside" and the same happens after \((1-t)\). Hence
\( f(B(a_{n-1})) = f(1 \cdot \ldots \cdot g) = g \) which is the generator of \( H_{2n-1}(g) \).

**Proposition 2.3.** Under the same hypothesis of Proposition 1, \( H_{2n}^g(A) \)
is again a free cyclic \( K \)-module generated by an element \( \beta \) which is
represented by a cocycle \( (\overline{a}_{2n+1}, \overline{a}_{2n}, \ldots, \overline{a}_0) \) such that \( \overline{a}_i \in A^i \) and
\( \overline{a}_{2n+1} \) has integral odd coefficient in \( g \cdot \ldots \cdot g \).

**Proof.** Since \( B: H_{2n-2}(g) \to H_{2n-1}(g) \) is surjective and \( H_{2n-2}^g \) is
cyclic generated by an element $\gamma$, $\text{Ker } B$ is again cyclic generated by $2\gamma$. Since 2 is not a zero divisor in $K$ the ideal $(2)$ in $K$ is again free.

Now $B(2a_{2n-1}) = 2h_1 \ast \ldots \ast g + \text{elements in Ker } f$ and $h \in Z$ (h odd). But $bB(a_{2n-1}) = 0$ because $bB(a_{2n-1}) = b(1-t) e_0 N(a_{2n-1}) = (1-t)b'e_0 N(a_{2n-1}) = (1-t)N(a_{2n-1}) = 0$ hence $2B(a_{2n-1})$ is a b-boundary, i.e. $2B(a_{2n-1}) = b(hg \ast \ldots \ast g + 1 \ast g \ast \ldots \ast g \ast 1 + \text{terms in ker } f)$.

(Lemma 1.1).

**THEOREM 2.4.** If 2 is not a zero divisor in $K$ the cyclic homology of $A = K[Z/2Z]$ is

$$
\text{HC}_{2n}(A) \cong K \otimes K
$$

$$
\text{HC}_{2n-1}(A) \cong K/2K \ast \ldots \ast K/2K \quad (n \text{ times}).
$$

**Proof.** According to Karoubi's result [K] and the previous comments it will be enough to prove: $\text{HC}^g_{2n}(A) = K$, $\text{HC}^g_{2n-1}(A) = 0$.

The value $\text{HC}^g_{2n}(A) = K$ follows by direct computations. According to proposition 2.3. $\text{HC}^g_{2n}(A) = 2$. $\text{HC}^g_{2n-2}(A)$ and from $\text{HC}^g_{2n-2}(A) \cong K$ follows $\text{HC}^g_{2n}(A) \cong K$ (since 2 is not a zero divisor in $K$).

For the odd dimensional homologies we use the exactness of

$$
\text{HC}^g_{2n-2}(A) \rightarrow \text{HC}^g_{2n-1}(A) \rightarrow \text{HC}^g_{2n-1}(A) \rightarrow \text{HC}^g_{2n-3}(A) \rightarrow 0
$$

and the fact that the first arrow is surjective (Prop.2.1).

Hence $\text{HC}^g_{2n-1}(A) \cong \text{HC}^g_{2n-3}(A)$ and for $n=1$, $\text{HC}^g_{-1}(A) = 0$ so the result follows by induction.

**LITERATURE**


Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires.

Recibido en noviembre de 1987.