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HOROSPHERES IN PSEUDO-SYMMETRIC SPACES

Jorge Vargas

Let G be a connected complex semisimple Lie group and G_o an inner real form of G. In this paper we study the space θ of all orbits in G/G_o of the totality of unipotent maximal subgroups of G.

INTRODUCTION

Let G, G_o, θ be as above. In this paper we provide a cross section of the action of G in θ (Theorem 4). We also prove that the orbits of a unipotent maximal subgroup of G in G/G_o are closed (Proposition 6) and an analogous of the Bruhat decomposition for G (Proposition 1).

STATEMENTS AND PROOFS

Let G be a complex, connected, semisimple, Lie group. G_o an inner real form of G, and B a Borel subgroup of G, such that $H_o = B \cap G_o$ is a compact, Cartan subgroup of G_o. Let H be the complexification of H_o. Lie groups will always be denoted by capital Roman letters. The corresponding Lie algebra will be denoted by the corresponding lower case german letter. The complexification of a real vector space, will be denote by adding the upperscript $\mathbf{C}$.

Let $\Phi(g,h)$ denote the root system of the pair (g,h). Fix K a maximal compact subgroup of G_o, such that $H_o \cap K$ is a maximal torus of K. K determines a Cartan decomposition of $g_o = k \oplus p$. If α is a root of the pair (g,h) its corresponding root space lies in $k^{\mathbf{C}}$ or $p^{\mathbf{C}}$. In the former case α is called compact and in the second case noncompact . Let σ or bar denote the conjugation of g with respect to g_o. Then for each α in $\Phi(g,h)$ it is possible to find root vectors Y_α, $Y_{-\alpha}$ such that:

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$Y_{\pm\alpha}$ lies in $k^{\mathbf{C}}$, $\sigma(Y_{\pm\alpha}) = -Y_{\mp\alpha}$, $[Y_\alpha, Y_{-\alpha}] = Z_\alpha$ and $\alpha(Z_\alpha) = 2$ for α compact.

$Y_{\pm\alpha}$ lies in $p^{\mathbf{C}}$, $\sigma(Y_{\pm\alpha}) = Y_{\mp\alpha}$, $[Y_\alpha, Y_{-\alpha}] = Z_\alpha$ and $\alpha(Z_\alpha) = 2$ for α noncompact.

Two roots are called strongly orthogonal if neither their sum nor their difference is a root. Let Ψ denote the system of positive roots in $\Phi(g,h)$ determined by the Lie algebra n of the unipotent radical N of B. For each noncompact root α in Ψ, let

$$c_\alpha = \exp(\tfrac{\pi}{4}(Y_{-\alpha} - Y_\alpha))$$

the inner automorphism associated to c_α is usually called "The Cayley transform associated α". In [S] is proved that if two positive noncompact roots are strongly orthogonal, then the associated Cayley transform commutes. Thus, if S denotes a subset of Ψ consisting of noncompact strongly orthogonal roots, then, the product

$$c_S = \prod_{\alpha \in S} c_\alpha$$

is well defined.

For any two Lie groups $G \supset H$, let

W(G,H) "The Weyl group of H in G"

denote the normalizer of H in G divided by H. Keeping in mind the notation written from the beginning we state and prove the first result of this paper.

PROPOSITION 1. $G = \cup\, G_o\, c_S\, w\, B$.

Here, the union runs over a set of representatives of W(G,H) and all the subsets S of Ψ, such that S consists of strongly orthogonal noncompact roots.

Proof. Since every Borel subgroup of G is equal to its own normalizer, the coset space G/B can be identified with the set of maximal solvable subgroups of G via the map $xB \to xBx^{-1}$. It follows that this map is equivariant. Now if B_1 is any Borel subgroup of G, B_1 contains a σ-invariant Cartan subgroup. Because $\sigma(B_1)$ is another Borel subgroup of G and by Bruhat's lemma $B_1 \cap \sigma(B_1)$ contains a Cartan subgroup. Fix a regular element h in $B_1 \cap \sigma(B_1)$ since, in [F] pag.479 is proved that for any regular element h, $zh + \bar{z}\sigma(h)$ is regular for suitable z in $\mathbf{C}$, we have that B_1 contains a σ-invariant regular element. Thus, B_1 contains a σ-invariant Cartan subgroup T. In [S] is proved that if T is any σ-invariant Cartan subgroup of G, there exists a strongly orthogonal subset S of the set of noncompact roots in Ψ, such that T is G_o-conjugated to $c_S H c_S^{-1}$. Therefore B_1 is G_o-conjugated to a Borel subgroup containing $c_S H c_S^{-1}$, for some strongly orthogonal set of noncompact roots in Ψ. Since any two Borel subgroups of G containing H are W(G,H) conjugated, we conclude that

$$B_1 = g\, c_S\, w\, B\, w^{-1} c_S^{-1} g^{-1} \qquad (g \in G_o,\ w \in W(G,H) \text{ and } c_S \text{ a Cayley transform})$$

Now, if x is in G, $B_1 = x B x^{-1}$ is a Borel subgroup, hence, via the map *between* G/B and the set of maximal solvable subgroups of G described above, we have that $x = g\, c_S\, w\, b$.

Q.E.D.

COROLLARY. If A is any Borel subgroup of G, then A contains a σ-invariant Cartan subgroup.

Towards the uniqueness of the decomposition in proposition 1 we prove

LEMMA 2. *Let a be any Borel subalgebra of g and h_1, h_2 two σ-invariant Cartan subalgebras of g contained in a. Then, there exists* $x \in A \cap G_o$ *such that* $h_2 = Ad(x) h_1$. *Here,* A *stands for Borel subgroup of* G*, corresponding to a.*

Proof. Let n be the nilpotent radical of a. Since $a = h_1 \oplus n$ and that a Cartan subalgebra of g, is a Cartan subalgebra of a ([F] 17.7), and any two Cartan subalgebras of a are A-conjugated ([F] 17.8) we have that $h_2 = Ad(n) h_1$, where n is an element of the unipotent radical of A. Because h_1 and h_2 are σ-invariant we have that

$$h_2 = \sigma(h_2) = \sigma(Ad(n)h_1) = Ad(\sigma(n))\sigma(h_1) = Ad(n)h_1$$

Hence, $Ad(n^{-1}\sigma(n))h_1 = h_1$, so if H_1 is the Lie group with Lie algebra h_1, we have that $n^{-1}\sigma(n) = w$ is in W(G,H).

If $z \in h_1$, in [F] is proved, for any n in the unipotent radical of a that

$$Ad(n) Z = Z + n(Z)$$

where n(Z) is an element of n, which depends on Z and n.

Therefore, for any $Z \in h_1$, since $n = \sigma(n) w$ we have that

$$Z + n_1(Z) = Ad(w) Z + n_2(Z)$$

where $n_1(Z)$ is in n and $n_2(Z)$ is an element of n plus its opposite, Lie algebra. Because g is the direct sum of a plus the opposite Lie algebra to n, we have that

$$\mathrm{Ad}(w)Z = Z \qquad \text{for any } Z \text{ in } h_1.$$

This allows us to conclude that

$$n = \sigma(n)h_o \qquad \text{with } h_o \text{ in } H_1.$$

The equality $h_o = \sigma(n)^{-1}n$ implies $\sigma(h_o) = h_o^{-1}$, and because H_1 is abelian connected and σ-invariant, we can find h_1 in H_1 such that

$$h_o = h_1^2 \quad ; \quad \sigma(h_1) = h_1^{-1}.$$

Let $n_1 = nh_1^{-1}$, then n_1 is in B. On the other hand,

$\sigma(n_1) = \sigma(n)\sigma(h_1^{-1}) = \sigma(n)h_1 = \sigma(n)h_oh_1^{-1} = \sigma(n)\sigma(n)^{-1}nh_1^{-1} = nh_1^{-1} = n_1.$

Thus n_1 is in $A \cap G_o$. Also $\mathrm{Ad}(n_1)h_1 = \mathrm{Ad}(n)\ \mathrm{Ad}(h_1^{-1})h_1 = \mathrm{Ad}(n)h_1 = h_2$.

Q.E.D.

COROLLARY. If A is any Borel subgroup of G and H_1, H_2 are σ-invariant Cartan subgroups of G which are in A, then H_1 is $G_o \cap A$ conjugated to H_2.

We keep the hypothesis and notation as in proposition 1 and lemma 2.

LEMMA 3. *We write*

$$G_o\, c_S\, w\, B = G_o\, c_{S'} w'\, B \qquad (*)$$

Here c_S, $c_{S'}$ *are Cayley transforms and* w, w' *are in* $W(G,H)$.

Then, the equality (*) *holds if and only if there exists* w_3 *in* $W(G_o,H)$ *such that* $w_3(S \cup (-S)) = S' \cup (-S')$ *and there exists* w *in* $W(G,H)$ *which is* c_S*-conjugated to an element of* $W(G_o,(c_SHc_S^{-1}) \cap G_o))$, *and if* w_4 *is in* $W(G,H)$ *satisfying*

$$c_{S'} = w_3\, c_S\, w_4\, c_S^{-1}$$

then $w' = w_3\, w_4\, w_5\, w$ *in* $W(G,H)$.

Proof. If the equality holds, then, there are g in G_o and b in B, such that

$$g\, c_S\, w\, b = c_{S'}\, w'$$

Hence, $A = g\, c_S\, w\, b\, B\, b^{-1}w^{-1}c_S^{-1}g^{-1} = c_{S'}\, w'\, B\, w'^{-1}\, c_{S'}^{-1}$, or

$$A = g\, c_S\, w\, B^{-1}w^{-1}c_S^{-1}g^{-1} = c_{S'}\, w'\, Bw'^{-1}\, c_S^{-1}.$$

Thus $g\, c_S\, w\, H\, w^{-1}c_S^{-1}g^{-1} = g\, c_S\, H\, c_S^{-1}g^{-1}$ and $c_{S'}\, w'\, H\, w'^{-1}\, c_{S'}^{-1} =$

$= c_{S'}\, H\, c_{S'}^{-1}$ are σ-invariant Cartan subgroups of A [S]. Because of lemma 2, there exists $b_1 \in A \cap G_o$ which carries $g\, c_S\, H\, c_S^{-1} g^{-1}$ onto $c_{S'}\, H\, c_{S'}$. Thus, $c_S\, H\, c_S^{-1}$ and $c_{S'}\, H\, c_{S'}^{-1}$ are G_o-conjugated. [S] implies that there exists $w_3 \in W(G_o,H)$ such that

$$w_3(S \cup (-S)) = S' \cup (-S').$$

Now, if β is any noncompact root c_β^2 is equal to "the reflection about β". Thus, c_β is equal to $c_{-\beta}$ times an element of $W(G,H)$. Moreover, in [V] is proven that, if $w \in W(G_o,H)$, then $c_{w\beta}$ is equal to $wc_\beta w^{-1}$ or $wc_\beta S_\beta w^{-1}$ (S_β = "reflection about β") depending on whether $Ad(w)Y_\beta = Y_{w(\beta)}$ or $Ad(w)Y_\beta = -Y_{w(\beta)}$.

Therefore, we conclude that the equality $gc_Swb = c_{S'}w'$ implies that there exist $w_3 \in W(G_o,H)$, w_4 product of reflections about roots in S, such that

$$g\, c_S\, w\, b = w_3 c_S w_4 w_3^{-1} w'$$

Set $g_1 = w_3^{-1} g$, and $w_6 = w_4 w_3^{-1} w'$ then we have that g_1 is in G_o, w_6 is in $W_{\mathbf{C}}$ and

$$g_1\, c_S\, w\, b = c_S\, w_6$$

Thus, $wbBb^{-1}w^{-1} = wBw^{-1}$ is a Borel subgroup containing $wHw^{-1} = H$, hence $(wBw^{-1}) \cap G_o = H_o$ (G_o is inner!).

Now, $c_S^{-1} g_1^{-1} c_S w_6 H w_6^{-1} c_S^{-1} g_1 c_S^{-1} = c_S^{-1} g_1^{-1} c_S H c_S^{-1} g_1 c_S = wbHb^{-1}w^{-1}$ is a σ-invariant Cartan subgroup of wBw^{-1}. By lemma 2, there exists h in $(wBw^{-1}) \cap G_o = H_o$ such that $wbHb^{-1}w^{-1} = hHh^{-1} = H$. Therefore, b lies in the normalizer of H in G and in B, which implies b is in H. Thus w and wb represent the same element of $W(G,H)$. Finally, let $w_5 = c_S^{-1} w_3^{-1} g c_S$. Because $w_5 = w_4 w_3^{-1} w' wb$, we have that $w_5 \in W(G,H)$. Hence w_5 is in $W(G,H) \cap c_S^{-1}\, W(G_o, (c_S H c_S^{-1}) \cap G_o) c_S$. In words, w_5 is conjugated to an element of the Weyl group of $c_S H c_S^{-1}$ in G_o.

Therefore we have proven

$$G_o c_S\, w\, B = G_o c_{S'} w' B \quad \text{implies that}$$

there are w_3 in $W(G_o,H_o)$, w_5 in $W(G,H)$ such that

$$w_3(S \cup (-S)) = S' \cup (-S')$$

w_5 is in $W(G,H)$ and is conjugated by c_S to an element of $W(G_o,(c_S H c_S^{-1}) \cap G_o)$; and if w_4 is in $W(G,H_o)$ such that

$$c_{S'} = w_3 c_S w_4 w_3^{-1}$$

then
$$w' = w_3 w_4 w_5 w\ .$$

Conversely. Let w, w_3 , w_4 , w_5 and w' as in the hypothesis of the lemma. Then

$$G_o c_{S'} w' B = G_o w_3 c_S w_4 w_3^{-1} w_3 w_4 w_5 w\ B =$$

$$= G_o c_S w_5 w\ B = G_o c_S w_5 c_S^{-1} c_S w\ B = G_o c_S w\ B.$$

Q.E.D.

Lemmas 2 and 3 allow us to parametrize in a useful manner the space of orbits of $G_o\backslash G$ by the action of the maximal unipotent subgroup of G.

Let N_1 be any maximal unipotent subgroup of G. The orbit of N_1 by $G_o x$ in $G_o\backslash G$ is the set $\{G_o x\ n\colon n \in N_1\}$.

Let θ be the set of all orbits of the totality of maximal unipotent subgroups of G. Since the conjugated of a maximal unipotent subgroup of G is a maximal unipotent subgroup of G, we have that G acts on θ by the rule

$$(G_o\ x\ N_1).g = G_o\ x\ g^{-1}\ (g\ N_1\ g^{-1}) \quad (x,\ g \text{ in } G).$$

From now on, we will only consider this action of G in θ. Let G_o, H_o, B as in the beginning of the paper. Let N be the unipotent radical of B. If N_1 is any maximal unipotent subgroup of G ([H]) there is g in G such that $N_1 = g\ N\ g^{-1}$. Thus, $G_o\ x\ N_1 = G_o\ x\ g\ N\ g^{-1} =$ $= (G_o\ x\ g\ N).g$.

Therefore we conclude:

Any element of θ is the translate by the action of G to an orbit of N (N being the unipotent radical of B).

Now, lemma 2 says that any N orbit is equal to an orbit of the type $G_o c_S w\ h\ N$ (where, $h \in H$, c_S is a Cayley transform and w is in W(G,H)). Thus, we have proved

THEOREM 4. *A family of representatives of the set* θ *of all the orbits of the totality of maximal unipotent subgroups of* G *in* $G_o\backslash G$ *by the action of* G *in* θ *is given by* $\{G_o c_S w\ h\ N\colon c_S \ldots,\ w \in W(G,H),\ h \in H\}$ *and* $G_o c_S w\ h\ N = G_o c_{S'} w' h' N$ *if and only if* S, S', w, w' *are related as in lemma 3.*

LEMMA 5. *Let* V *be a real finite dimensional vector space and* N *a unipotent subgroup of* $G\ell(V)$. *Let* $V_{\mathbf{C}}$ *be a complexification of* V *and* $N^{\mathbf{C}}$ *the Zariski closure of* N *in* $G\ell(V_{\mathbf{C}})$ *(we think of* $G\ell(V)$ *included in* $G\ell(V_{\mathbf{C}})$ *in the usual way). Then*

i) *For every* x *in* V, $N^{\mathbf{C}}.x$ *is equal to the Zariski closure of* N.x.

ii) $(N^{\mathbf{C}}.x) \cap V = N.x$.

Proof. Since $N^{\mathbf{C}}$ is a unipotent subgroup of $G\ell(V_{\mathbf{C}})$, we have that $N^{\mathbf{C}} x$ is closed in $V_{\mathbf{C}}$ [H], therefore $N^{\mathbf{C}} x$ contains the Zariski closure of $N.x$. On the other hand, the map $T \to T(x)$ is a polynomial map from $N^{\mathbf{C}}$ to $V_{\mathbf{C}}$, hence, it is continuous if we set the Zariski topology in both $N^{\mathbf{C}}$ and $V_{\mathbf{C}}$.

Besides in [B] is proved that the Zariski closure of N is $N^{\mathbf{C}}$, thus $N^{\mathbf{C}} x$ is contained in the Zariski closure of $N.x$, and we have proved i).

In order to prove ii) we need to verify that $(N^{\mathbf{C}} x) \cap V$ is contained in $N.x$. We do it by induction on dimension of V. If $\dim V = 1$, the unipotent subgroup of $G\ell(V)$ is $\{1\}$.

If $\dim V > 1$. Since, N is a unipotent subgroup of $G\ell(V)$, Engel's theorem implies that there exists a non zero v in V such that $n(v) = v$ for every n in N.

Since $N^{\mathbf{C}}$ is the Zariski closure of N, we have that $n(v) = v$ for every n in $N^{\mathbf{C}}$. By the inductive hypothesis, we conclude that if T is in $N^{\mathbf{C}}$, a in V, c in $\mathbf{C}$ and $Tx = a + cv$, then there exists S in N such that $Tx = Sx + dv$, (d in $\mathbf{C}$).

Now, let T be in $N^{\mathbf{C}}$, such that Tx belongs to V. Owing to the inductive hypothesis, there exist S in N, d in $\mathbf{C}$ such that $Tx = Sx + dv$. Since Tx and Sx belong to V, we have that d is real. If $d = 0$, we are done.

If $d \neq 0$, let M be $M = \{n \text{ in } N^{\mathbf{C}}: n(x) \equiv x \ (\mathbf{C}_v)\}$. It is clear that M is a Zariski closed subgroup of $N^{\mathbf{C}}$ and that $S^{-1}T$ belongs to M ($S^{-1}T(x) = S^{-1}(Sx + dv) = x + dS^{-1}(v) = x + dv$, $S^{-1}(v) = v$!!!). Since x and v are in V, it follows that M is invariant under the conjugation of $N^{\mathbf{C}}$ with respect to N. Therefore M has a real form M_1. In other words , $M_1 = M \cap N$ is a real form of M. Now the map $n \to n(x) - x$ from M into $\mathbf{C}v$ is non constant, because $S^{-1}T$ goes to dv_1, which is nonzero. Besides it is a polynomial map. Since the unique non trivial Zariski closed subgroup of $\mathbf{C}v$ is itself, we have that the map $n \to n(x)-x$ is onto. Since, for n in M_1, $n(x)-x$ is a real multiple of v we conclude that there exists R in N such that $R(x)-x = -dv$ (d is real!!).

Therefore $-dv = S^{-1}Tx-x = R(x)-x$, hence $Tx = SR(x)$. Since SR belongs to N we conclude the proof of the lemma.

PROPOSITION 6. *Let* N *be any maximal unipotent subgroup of* G. *Then the orbit* G_o x N *of* $G_o x$ *by* N *in* $G_o\backslash G$ *is closed in* $G_o\backslash G$.

Proof. Think of G as a real Lie group and let $G^{\mathbf{C}}$ be its complexification. Since G is a linear Lie group ([W] Wallach) G is contained in $G^{\mathbf{C}}$. Let $G_o^{\mathbf{C}}$ be the complexification of G_o in $G^{\mathbf{C}}$. Since $G^{\mathbf{C}}$ and $G_o^{\mathbf{C}}$ are semisimple Lie groups, the complex homogeneous manifold $G_o^{\mathbf{C}}\backslash G^{\mathbf{C}}$ is a non singular affine variety. Since $G_o^{\mathbf{C}} \cap G = G_o$, we have that $G_o\backslash G$ is a real submanifold of $G_o^{\mathbf{C}}\backslash G^{\mathbf{C}}$. Let $N_1^{\mathbf{C}}$ be the complexification of N_1 in $G^{\mathbf{C}}$. Then ([H], page 125) the orbit $G_o^{\mathbf{C}}$ x $N_1^{\mathbf{C}}$ is closed in $G_o^{\mathbf{C}}\backslash G^{\mathbf{C}}$. Since, for x in G, the orbit G_o x N_1 is equal to $(G_o^{\mathbf{C}}\ x\ N_1^{\mathbf{C}}) \cap G$, we have that the orbit G_o x N_1 is closed in $G_o\backslash G$.

PROPOSITION. *Let* N_1 *be any maximal unipotent subgroup of* G *and let* $G_o x\ N_1$ *be an orbit of* N_1 *in* $G_o\backslash G$. *Let* σ_x *be the conjugation of* G *with respect to the real form* $x^{-1}G_o\ x$. *Then:* 1) *The isotropy subgroup of* N_1 *at* $G_o x$ *is the real form of* $N_1 \cap \sigma_x(N_1)$ *determined by* σ_x. 2) *The isotropy subgroup of* N_1 *at* $G_o x$ *is connected.*

Proof.
$$\{n \in N_1 : G_o x\ n = G_o x\} = \{n \in N_1 : x\ n\ x^{-1} \in G_o\} =$$
$$= \{n \in N_1 : n \in x^{-1}G_o x\} = N_1 \cap (x^{-1}G_o x).$$

Thus, if $n \in N_1$ and $G_o x\ n = G_o\ x$, then $\sigma_x(n) = n$.
Hence $\sigma_x(N_1 \cap (x^{-1}G_o x)) = N_1 \cap (x^{-1}\ G_o x)$. Therefore
$$N_1 \cap (x^{-1}G_o x) = (N_1 \cap (x^{-1}G_o x)) \cap (\sigma_x(N_1 \cap (x^{-1}G_o x)) =$$
$$= N_1 \cap (x^{-1}G_o x) \cap \sigma_x(N_1) \cap (x^{-1}G_o x) = (N_1 \cap \sigma_x(N_1)) \cap (x^{-1}G_o x).$$

Which proves 1. Let's prove the second affirmation. Since, [H], $N_1 \cap \sigma_x(N_1)$ is a unipotent algebric group, it is connected. Moreover, because of a theorem of [B], the group of real points of the algebraic group $N_1 \cap \sigma_x(N_1)$ has finitely many connected components. Hence, if n belongs to $N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_o x)$, then some power is in the connected component of $N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_o x)$.
Say x^k is in the connected component of $N_1 \cap (\sigma_x(N_1)) \cap (x^{-1}G_o x)$. Since the exponential map on any real nilpotent connected group is onto, there exists y in the Lie algebra of $N_1 \cap \sigma_x(N) \cap (x^{-1}G_o x)$ such that $x^k = \exp(y)$. On the other hand, because of Engels theo-

rem and [F] any unipotent algebraic subgroup of $G\ell(n,C)$ is simply connected, and hence, [F] the exponential map of $N_1 \cap \sigma_x(N_1)$ is bijective. Thus, the equality $x^k = \exp(y) = (\exp(1/k\ y))^k$ implies that $x = \exp(1/k\ y)$. Hence the group $N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_o x)$ is connected. Q.E.D.

The following fact is useful.

PROPOSITION. *Let* K *be a complex Lie group, such that the exponential map of* K *is bijective (for example,* K *unipotent and connected). Let* σ *be an involutive real automorphism of* K*; let* K_o *be the fixed point set of* σ *and* K_- *the subset of those elements of* G *that are taken by* σ *into its inverse. Then* $K = K_o K_-$.

Proof. We want to prove that for a given y in K, there exist x in K_o and z in K_- such that $y = xz$.
Let b be $b = \sigma(y)^{-1}y$. It is clear that $\sigma(b) = b^{-1}$. Since the exponential map is onto, there exists Y in k such that $b = \exp(Y)$.
Since $\sigma(b) = \exp(\sigma Y) = b^{-1} = \exp(-Y)$, and the exponential map is injective, we have that $\sigma Y = -Y$. Thus $z = \exp(1/2\ Y)$ belongs to K_-. Let $x = yz^{-1}$. Then $\sigma(x) = \sigma(y)\sigma(z^{-1}) = \sigma(y)\sigma(z)^{-1} =$
$= yb^{-1}\sigma(z)^{-1} = yb^{-1}z = yz^{-1} = x$.
Q.E.D.

PROPOSITION. *Let* $B \subset G$ *be any Borel subgroup* (G *as usual*). *Let* H *be a* σ*-invariant, Cartan subgroup of* B. *Let* H_o *be the set of real points of* H. *Then* $B \cap G_o = H_o\ (N \cap G_o)$ (N *being the unipotent radical of* B).

Proof. If hn belongs to $B \cap G_o$ then $hn = \sigma(hn) = \sigma(h)\sigma(n)$. Therefore $\sigma(n) = \sigma(h)^{-1}hn$ belongs to B. Since H is σ-invariant, we have that $\sigma(h)^{-1}h$ is in H. Thus (the decomposition $B = H\ N$) says that $n = \sigma(n)$ and $\sigma(h)^{-1}h = 1$. Hence $\sigma(h) = h$.
Q.E.D.

The next step is to compute the normalizer of an orbit of N in G/G_o. Because of the equality $N\ x\ G_o = x(x^{-1}Nx).G_o$, we have that any orbit in G/G_o is the translate of an orbit through the coset G_o. Thus, we conclude.
The normalizer of any N-orbit in G/G_o is conjugated (in G) to the

normalizer of an orbit of the type N.O (O = coset G_o).

Now for a fixed unipotent maximal subgroup N of G, if B denotes the unique Borel subgroup containing N, it is clear that $(B \cap G_o)$ N normalizes the orbit N.O. We would like to prove the equality. We have been able to prove this only in particular cases.

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FAMAF-CIEM
Ciudad Universitaria
5000 Córdoba, Argentina

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