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STURM-LIOUVILLE PROBLEMS FOR THE SECOND ORDER EULER OPERATOR DIFFERENTIAL EQUATION

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SUMMARY. By means of the reduction to an algebraic operator problem, explicit expressions for solutions of Sturm-Liouville problems for the second order Euler operator differential equation are given.

1. INTRODUCTION.

The classical theory of ordinary differential equations [2,6], studies the eigenvalue problem and the expressions of solutions for Sturm-Liouville problems of the type

$$\begin{aligned} (p(t)x')'+\lambda(r(t)-q(t))x &= 0 \\ a_1x(a)+a_2x'(a) &= 0 \\ b_1x(b)+b_2x'(b) &= 0 \\ 0 < a \leqslant t &\leqslant b \end{aligned}$$

where λ is a real parameter and p,q and r are continuous functions on the interval [a,b]. Second order differential equations in Hilbert space occur frequently in vibrational systems [5,9], statistical physics [17], etc. These equations have been studied by several authors with different techniques [3-4,8,10,12,13,17].

In this paper we consider the Sturm-Liouville problem

$$\begin{aligned} t^2X^{(2)}(t)+A_1tX^{(1)}(t)+\lambda A_oX(t) &= 0 \\ E_1X(a)+E_2X^{(1)}(a) &= 0 \\ F_1X(b)+F_2X^{(1)}(b) &= 0 \\ 0 < a \leqslant t &\leqslant b \end{aligned} \tag{1.1}$$

where X(t), A_j, for j = 0,1, E_i,F_i, for i = 1,2, are bounded li-

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near operators on a complex separable Hilbert space H and λ is a complex parameter. We are interested in finding existence conditions and explicit expressions of solutions for the problem (1.1). Some analogous problems to (1.1) have been studied with different techniques in [11].

Throughout this paper L(H) denotes the algebra of all bounded linear operators on H. If T lies in L(H), its spectrum $\sigma(T)$ is the set of all complex numbers z such that zI-T is not invertible in L(H). The point spectrum of T, denoted $\sigma_p(T)$ is the set of all complex numbers z such that zI-T is not injective. From [1,p.240], it follows that z lies in $\sigma_p(T)$, if and only if, zI-T is a left divisor of zero in L(H), this is, there exists a nonzero operator S in L(H) such that (zI-T)S = 0.

Finally we recall that if W is an operator in L(H) with a closed range, then the orthogonal generalized inverse of W is a bounded operator in L(H) denoted by W^+, see [14,p.60-65].

2. EXPLICIT SOLUTIONS

Let us consider the second order Euler operator differential equation

$$t^2X^{(2)}(t)+tA_1X^{(1)}(t)+\lambda A_oX(t) = 0 \qquad (1.2)$$

Making the change of variable t = exp(u), the equation (1.2) is reduced to an equivalent one with independent variable u. This new equation takes the form

$$\ddot{X}+(A_1-I)\dot{X}+\lambda A_oX = 0 \qquad (2.2)$$

In an analogous way to the scalar case we can obtain a pair of solutions of equation (2.2) from a pair of solutions of the associated algebraic operator equation

$$U^2+(A_1-I)U+\lambda A_o = 0 \qquad (3.2)$$

It is clear that if X_i, for i = 0,1, are solutions of equation (3.2), then $Z_i(t) = \exp(tX_i)$, are solutions of equation (2.2).

The resolution problem of equation (3.2) is closely related to the problem of the linear factorization of the polynomial operator $L(z) = z^2+(A_1-I)z+\lambda A_o$. In fact, equation (3.2) is solvable, if and only if, L(z) admits a linear factorization [5]. If H is a finite-dimensional space, it occurs if the companion operator

$$C_L = \begin{bmatrix} 0 & I \\ -\lambda A_o & I-A_1 \end{bmatrix}$$

is diagonalizable, [15]. For instance, if the eigenvalues of C_L are simple, that is, when the Jordan matrix of C_L is diagonal. In [7] it is proved that if H is infinite-dimensional a lot of equations of the type (3.2) are solvable. However, equation (3.2) can be unsolvable, for instance, if $A_1 = I$ and λA_o is an unilateral weighted shift operator, equation (3.2) is unsolvable, [16,p.63]. A methodology for solving equation (3.2) by reduction to an easier equation of first order is given in [7].

By reduction to a first order extended linear system on $H \oplus H$ in the natural way, it is well known that a Cauchy problem for equation (2.2) has only one solution. The following result permits to express any solution of (2.2) in terms of the exponential functions $Z_i(t) = \exp(tX_i)$, for $i = 0,1$. Notice that it is like the scalar case, but we need to impose certain conditions to the solutions X_i of the algebraic operator equation (3.2).

LEMMA 1. *Let us consider the operator differential equation* (2.2) *where* X_o, X_1, *are solutions of equation* (3.2) *such that* $X_1 - X_o$ *is invertible in* L(H), *then any solution* X(t) *of equation* (2.2) *is uniquely expressed in the form*

$$X(t) = Z_o(t)C+Z_1(t)D \tag{4.2}$$

where the operators C,D *are given by the expressions*

$$\begin{aligned} C &= \exp(-aX_o)\{C_o+(X_1-X_o)^{-1}(X_oC_o-C_1)\} \\ D &= \exp(-aX_1)\{X_1-X_o)^{-1}(C_1-X_oC_o)\} \end{aligned} \tag{5.2}$$

where

$$C_o = X(a) \quad , \quad C_1 = X^{(1)}(a).$$

Proof. It is clear that for any operators C and D, the operator function X given by (4.2) satisfies the equation (2.2). From the uniqueness of solutions for a Cauchy problem related to this equation [8], in order to prove the lemma we must show that given a solution X of (2.2), there is a unique pair of operators C and D such that the representation (4.2) is available.

By differentiation in (4.2) it follows that $X^{(1)}(t) = Z_o(t)X_oC + Z_1(t)X_1D$, thus taking into account the initial conditions, the operators C and D must verify

$$C_o = X(a) = \exp(aX_o)C+\exp(aX_1)D$$

$$C_1 = X^{(1)}(a) = \exp(aX_o)X_oC+\exp(aX_1)X_1D$$

or equivalently

$$\begin{pmatrix} \exp(aX_o) & \exp(aX_1) \\ X_o\exp(aX_o) & X_1\exp(aX_1) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} C_o \\ C_1 \end{pmatrix} \tag{6.2}$$

Let S be the operator matrix appearing in the left hand side of equation (6.2). It is easy to show that S is invertible and that S^{-1} is given by the operator matrix

$$S^{-1} = \begin{pmatrix} \exp(-aX_o)\{I+(X_1-X_o)^{-1}X_o\} & -\exp(-aX_o)(X_1-X_o)^{-1} \\ -\exp(-aX_1)(X_1-X_o)^{-1}X_o & \exp(-aX_1)(X_1-X_o)^{-1} \end{pmatrix}$$

Hence the result is established.

The following result characterizes the existence of nontrivial solutions of the boundary value problem (1.1) and it yields an explicit expression of solutions when they exist.

THEOREM 2. *Let us consider the boundary value problem* (1.1) *and let* X_o,X_1 *be solutions of* (3.2) *such that* X_1-X_o *is invertible. Let* W *be the following operator matrix*

$$W = \begin{pmatrix} (E_1+E_2X_o)\exp(aX_o) & (E_1+E_2X_1)\exp(aX_1) \\ (F_1+F_2X_o)\exp(bX_o) & (F_1+F_2X_1)\exp(bX_1) \end{pmatrix} \tag{7.2}$$

Then

(i) *The only solution of problem* (1.1) *is the trivial one* $X(t) = 0$, *if and only if* $0 \notin \sigma_p(W)$.

(ii) *If the operator* $E_1+E_2X_o$ *is invertible and we define the operator*
$V = (F_1+F_2X_1)\exp(bX_1)-(F_1+F_2X_o)\exp((b-a)X_o)(E_1+E_2X_o)^{-1}(E_1+E_2X_1)\exp(aX_1)$
then the condition $0 \notin \sigma_p(V)$ *is equivalent to the condition* $0 \notin \sigma_p(W)$ *expressed in* (i).

(iii) *If* H *is finite-dimensional, there are nontrivial solutions of* (1.1), *if and only if* W *is singular.*

(iv) *If* W *has a closed range and* W^+ *denotes its orthogonal generalized inverse, then the general solution of* (1.1) *is given by* (4.2) *where* C,D *are given by the expression*

$$\begin{pmatrix} C \\ D \end{pmatrix} = (I - W^{+}W)Z \ ,$$

where I *is the identity operator in* $L(H \oplus H)$ *and* Z *is an arbitrary operator of the form* $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ *, with* $Z_i \in L(H)$, *for* $i = 1,2$.

Proof. (i) From lemma 1, it is sufficient to find operators C,D, non simultaneously zero, such that the operator function defined by (4.2) satisfies the boundary value conditions appearing in (1.1). Notice that this is equivalent to the existence of an nonzero solution (C,D) of the algebraic system

$$W \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8.2}$$

From (8.2), there are operators C,D non simultaneously zero which satisfy this equation, if and only if, W is a left divisor of zero, this is, $0 \notin \sigma_p(W)$.

(ii) Let $W = (W_{ij})$, for $1 \leq i, j \leq 2$, the operator matrix defined by (7.2). From the hypothesis, the operator $W_{11} = (E_1 + E_2 X_o)\exp(aX_o)$ is invertible. Thus we can decompose W in the following way

$$W = \begin{pmatrix} I & 0 \\ W_{21}W_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} - W_{21}W_{11}^{-1}W_{12} \end{pmatrix} \begin{pmatrix} I & W_{11}^{-1}W_{12} \\ 0 & I \end{pmatrix} \tag{9.2}$$

It is clear that the first and the third factor in the right hand side of (9.2) are invertible operators on $H \oplus H$. From here the condition $0 \notin \sigma_p(W)$ is equivalent to the condition $0 \notin \sigma_p(W_{22} - W_{21}W_{11}^{-1}W_{12})$. Notice that $W_{22} - W_{21}W_{11}^{-1}W_{12}$ is the operator V given in (ii).

(iii) If H is finite-dimensional then the point spectrum $\sigma_p(W)$ coincides with the spectrum $\sigma(W)$.

(iv) The result is a consequence of proposition (1.4) of [14,p.8] and from [14,p.60-65].

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