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MIXED ELLIPTIC PROBLEMS WITH SOLUTIONS OF NON-CONSTANT SIGN

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ABSTRACT: We study some mixed elliptic differential problems with phase-change, i.e. with solutions of non-constant sign as functions of the Dirichlet and Neumann data.

KEY WORDS: Stefan problem, free boundary problems, phase-change problems, variational inequalities, optimization problems, Mixed elliptic problem.

I. INTRODUCTION

We consider a heat conducting material occuping Ω, a bounded domain of $\mathbb{R}^n$ ($n = 1, 2, 3$ in practice), with a sufficiently regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (with $\text{meas}(\Gamma_1) \equiv |\Gamma_1| > 0$, $|\Gamma_2| > 0$ and $|\Gamma_3| \geq 0$) . We assume, without loss of generality, that the phase-change temperature is 0°C. We impose a temperature $b = b(x) > 0$ on Γ_1 and an outcoming heat flux $q = q(x) > 0$ on Γ_2 ; we also suppose that the portion of the boundary Γ_3 (when it exists) is a wall impermeable to heat, i.e. the heat flux on Γ_3 is null. If we consider in Ω a steady-state heat conduction problem, then we are interested in finding sufficient and/or necessary conditions for the heat flux q on Γ_2 to obtain a change of phase in Ω, that is, a steady-state two-phase Stefan problem in Ω. Following [Ta1] we study the temperature $\theta = \theta(x)$, defined for $x \in \Omega$. The set Ω can be expressed in the form :

$$\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} \ . \tag{1}$$

where

$$\begin{aligned} \Omega_1 &= \left\{ x \in \Omega \ / \ \theta(x) < 0 \right\} , \\ \Omega_2 &= \left\{ x \in \Omega \ / \ \theta(x) > 0 \right\} , \\ \mathcal{L} &= \left\{ x \in \Omega \ / \ \theta(x) = 0 \right\} , \end{aligned} \tag{2}$$

are the solid phase, the liquid phase and the free boundary (e.g. a surface in $\mathbb{R}^3$) that separates them respectively. The temperature θ can be represented in Ω in the following way :

$$\theta(x) = \begin{cases} \theta_1(x) < 0 , & x \in \Omega_1 , \\ 0 , & x \in \mathcal{L} , \\ \theta_2(x) > 0 , & x \in \Omega_2 , \end{cases} \tag{3}$$

and satisfies the conditions below :

$$\begin{aligned} &\text{i) } \Delta\theta_i = 0 \quad \text{in } \Omega_i \ (i = 1, 2) , \\ &\text{ii) } \theta_1 = \theta_2 = 0 , \ k_1 \frac{\partial\theta_1}{\partial n} = k_2 \frac{\partial\theta_2}{\partial n} \quad \text{on } \mathcal{L} , \\ &\text{iii) } \theta_2 \mid_{\Gamma_1} = b , \qquad \text{iv) } \frac{\partial\theta}{\partial n} \mid_{\Gamma_3} = 0 , \\ &\text{v) } \begin{cases} - k_2 \frac{\partial\theta_2}{\partial n} \mid_{\Gamma_2} = q & \text{if } \theta \mid_{\Gamma_2} > 0 , \\ - k_1 \frac{\partial\theta_1}{\partial n} \mid_{\Gamma_2} = q & \text{if } \theta \mid_{\Gamma_2} < 0 , \end{cases} \end{aligned} \tag{4}$$

where $k_i > 0$ is the thermal conductivity of phase i (i = 1 : solid phase, i = 2 : liquid phase), $b > 0$ is the temperature given on Γ_1 , and $q > 0$ is the heat flux given on Γ_2 .

Problem (4) represents a free boundary elliptic problem (when $\mathcal{L} \neq \emptyset$) where the free boundary $\mathcal{L}$ (unknown a priori) is characterized by the three conditions (4ii). Following the idea of [Ba, Du1, Du2, Fre, Ta1] we shall transform (4) into a new elliptic problem but now without a free boundary. If we define the function u in Ω as follows

$$u = k_2\, \theta^+ - k_1\, \theta^- \quad \left(\theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \right) \quad \text{in } \Omega , \tag{5}$$

where θ^+ and θ^- represent the positive and the negative parts of the function θ respectively, then problem (4) is transformed into

$$\text{(6)}\qquad \begin{aligned} &\text{i) } \Delta u = 0 \quad \text{in } D'(\Omega), \\ &\text{ii) } u \mid_{\Gamma_1} = B \ , \ B = k_2 b > 0 \ , \\ &\text{iii) } -\frac{\partial u}{\partial n} \mid_{\Gamma_2} = q \ , \ \frac{\partial u}{\partial n} \mid_{\Gamma_3} = 0 \ , \end{aligned}$$

whose variational formulation is given by

$$\text{(7)}\qquad a(u,v-u) = L(v-u) \quad , \quad \forall\, v \in K \ , \ u \in K \ ,$$

where

$$\text{(8)}\qquad \begin{aligned} &V = H^1(\Omega) \quad , \quad V_0 = \left\{ v \in V \,/\, v \mid_{\Gamma_1} = 0 \right\} , \\ &K = K_B = \left\{ v \in V \,/\, v \mid_{\Gamma_1} = B \right\} , \end{aligned}$$

$$a(u,v) = \int_\Omega \nabla u \,.\, \nabla u \; dx \ , \quad L(v) = L_q(v) = -\int_{\Gamma_2} q\, v \; d\gamma \ .$$

Under the hypotheses $L \in V_0'$ (e.g. $q \in L^2(\Gamma_2)$) and $B \in H^{1/2}(\Gamma_1)$, there exists a unique solution of (7) which is characterized by the following minimization problem [BC, Du3, KS, Ro, Ta3]

$$\text{(9)}\qquad J(u) \le J(v) \quad , \quad \forall\, v \in K \ , \ u \in K \ ,$$

where

$$\text{(10)}\qquad J(v) = J_q(v) = \tfrac{1}{2}\, a(v,v) - L(v) = \tfrac{1}{2}\, a(v,v) + \int_{\Gamma_2} q\, v\, d\gamma \ .$$

LEMMA 1: If $u = u_{qB}$ is the unique solution of problem (7) for data q on Γ_2 and $B > 0$ on Γ_1, then we have the monotony property :

$$\text{(11)}\qquad B_1 \le B_2 \text{ on } \Gamma_1 \text{ and } q_2 \le q_1 \text{ on } \Gamma_2 \;\Rightarrow\; u_{q_1B_1} \le u_{q_2B_2} \text{ in } \bar{\Omega} \ .$$

Moreover,

$$\text{(12)}\qquad q > 0 \text{ on } \Gamma_2 \quad \Rightarrow \quad u_{qB} \le \underset{\Gamma_1}{\mathrm{Max}}\; B \ \text{ in } \bar{\Omega} \ ,$$

and function $u = u_{qB}$ satisfies the equality

$$\text{(13)}\qquad a(u^-,u^-) = \int_{\Gamma_2} q\, u^-\, d\gamma \ .$$

Proof. To prove (11) we shall take into account the following equivalence ($u_i = u_{q_iB_i}$, $i=1,2$) :

(14) $$u_1 \le u_2 \text{ in } \overline{\Omega} \Leftrightarrow w = 0 \text{ in } \overline{\Omega} \ ,$$

where

(15) $$w = (u_2 - u_1)^- \ .$$

Since $w \in V_0$, then, if we use $v = u_2 + w \in K_{B_2}$ in the variational equality (7) corresponding to u_1, and $v = u_1 + w \in K_{B_1}$ in the one corresponding to u_2 and we later subtract them, we have

(16) $$0 \le \int_{\Gamma_2} (q_1 - q_2)\, w \, d\gamma = a(u_2 - u_1, w) = -\, a(w,w) \le 0 \ ,$$

that is, $w = 0$ in Ω.

We prove (12) in a similar way. Moreover, it is enough to choose $v = u^+ \in K$ in (7) to obtain (13).

COROLLARY 2. From (13), we deduce

(17) $$u^- \neq 0 \text{ in } \overline{\Omega} \Leftrightarrow u^- \neq 0 \text{ on } \Gamma_2 \ ,$$

where $q > 0$ and $B > 0$.

In paragraph II. we shall consider three problems (Problem 1 to 3) related to (6) or (7).

Now, we replace the condition (4iii) by the following one [Ta1] :

(18) $$\begin{aligned} -k_2 \frac{\partial \theta_2}{\partial n} \Big|_{\Gamma_1} &= \alpha\,(k_2\,\theta_2 - B) \qquad \text{if} \quad \theta|_{\Gamma_1} > 0 \ , \\ -k_1 \frac{\partial \theta_1}{\partial n} \Big|_{\Gamma_1} &= \alpha\,(k_1\,\theta_1 - B) \qquad \text{if} \quad \theta|_{\Gamma_1} < 0 \ , \end{aligned}$$

where α = const. > 0 represents a heat transfer coefficient on Γ_1. We are interested in studying the temperature $\theta = \theta_\alpha$, represented in Ω by (3), which satisfies the conditions

(19) (4i,ii,iv,v) and (18) .

If we define the function u_α in Ω by (5), then problem (19) is transformed into

(20) $$\begin{aligned} &\text{i)}\ \Delta u = 0 \quad \text{in } D'(\Omega), \\ &\text{ii)}\ -\frac{\partial u}{\partial n}\Big|_{\Gamma_1} = \alpha\,(u - B)\ ,\ B = k_2\, b > 0\ , \\ &\text{iii)}\ -\frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q \ ,\quad \frac{\partial u}{\partial n}\Big|_{\Gamma_3} = 0 \ , \end{aligned}$$

whose variational formulation is given by ($u = u_{\alpha qB}$) :

(21) $$a_\alpha(u,v) = L_{\alpha qB}(v) \,,\ \forall\, v \in V \,,\ u \in V \,,$$

where

(22) $$a_\alpha(u,v) = a(u,v) + \alpha \int_{\Gamma_1} u\, v\; d\gamma \,,$$
$$L_{\alpha qB}(v) = L_q(v) + \alpha \int_{\Gamma_1} B\, v\; d\gamma \,.$$

Under the hypotheses $L_{\alpha qB} \in V'$ (e.g. $q \in L^2(\Gamma_2)$ and $B \in H^{1/2}(\Gamma_1)$), there exists a unique solution of (21) which is characterized by the following minimization problem [BC, Du3, KS, Ro, Ta3]:

(23) $$G(u) \le G(v) \quad,\quad \forall\, v \in V \quad,\quad u \in V \quad,$$

where

(24) $$G(v) = G_{\alpha qB}(v) = \tfrac{1}{2}\, a_\alpha(v,v) - L_{\alpha qB}(v) = J_q(v) + \tfrac{\alpha}{2} \int_{\Gamma_1} v^2\, d\gamma - \alpha \int_{\Gamma_1} Bv\, d\gamma \,.$$

LEMMA 3: If $u = u_{\alpha qB}$ is the solution of problem (21) for data $q > 0$ on Γ_2, $B > 0$ on Γ_1 and $\alpha > 0$, then we have the following properties (for a given $B > 0$):

(25)
(i) $u_{\alpha qB} \le B$ in Ω , $\forall\, \alpha > 0$, $\forall\, q > 0$,
(ii) $u_{\alpha qB} \le u_{qB} \le B$ in Ω , $\forall\, \alpha > 0$, $\forall\, q > 0$,
(iii) $u_{\alpha_1 q_1 B} \le u_{\alpha_2 q_2 B}$ in Ω , $\forall\, \alpha_1 \le \alpha_2$, $\forall\, q_2 \le q_1$,
(iv) $M_2 \le u_{\alpha qB} \le M_1$ in Ω , $\forall\, \alpha > 0$, $\forall\, q > 0$,

where

(26) $$M_2 = M_2(\alpha,q,B) = \min_{\Gamma_2} u_{\alpha qB} \,, \qquad M_1 = M_1(\alpha,q,B) = \max_{\Gamma_1} u_{\alpha qB} \,.$$

Moreover, we have that

(27) $$\lim_{\alpha \to +\infty} u_{\alpha qB} = u_{\alpha q} \quad \text{strongly in } V \,,$$

where $u_{\alpha q}$ is the solution of (7).

Proof. We use a similar method to the one developed in Lemma 1 taking into account that the bilinear form a_1 is coercive on V, i.e. [KS, Ta1, Ta3]

(28) $$\exists\, \lambda_1 > 0 \;/\; a_1(v,v) = a(v,v) + \int_{\Gamma_1} v^2\, d\gamma \ge \lambda_1\, \|v\|_V^{\,2} \,,\ \forall\, v \in V \,.$$

Moreover , so it is the bilinear form a_α and we have

(29) $a_\alpha(v,v) \geq \lambda_\alpha \|v\|_V^2$, $\forall$ v $\in$ V , $\lambda_\alpha = \lambda_1 \, Min(1,\alpha)$.

COROLLARY 4 : From (25), we deduce

(30) $\underset{\bar{\Omega}}{Max}\, u_{\alpha qB} = M_1$, $\underset{\bar{\Omega}}{Min}\, u_{\alpha qB} = M_2$.

where the elements M_1 and M_2 are defined in (26).

In paragraph II. we shall consider a problem (Problem 4) related to (20) or (21).

NOTE 1. Many others free boundary problems for elliptic or parabolic partial differential equations (of Stefan type) can be found in [BC, CJ, Cr, Di, Du2, EO, Fri, Li, Ma2, Pr, Ro, Ru, Ta3, Ta4, Ta6].

NOTE 2. We shall denote by (N−n) the formula (n) of Section N and we shall omit N in the same paragraph. Idem for theorems, lemmas, corolaries, remarks and notes.

We shall also omit the space variable x $\in$ Ω for every function defined in Ω.

II. ELLIPTIC DIFFERENTIAL PROBLEMS WITH OR WITHOUT PHASE CHANGE.

We shall give four problems, with their corresponding solutions, which are related to mixed elliptic partial differential equations (three of them are related to problem (I-6) or (I-7) and one of them is related to problem (I-20) or (I-21)).

Problem 1: For the constant case B > 0 and q > 0 , find a constant $q_0 = q_0(B) > 0$ such that for q > $q_0(B)$ we have a steady-state two-phase Stefan problem in Ω, that is the solution u of (I-7) is a function of non-constant sign in Ω.

Remark 1: From (I-17) we deduce that an answer to problem 1 is the element q for which u takes negative values on the boundary Γ_2 .

LEMMA 1: Let $u = u_q$ be the unique solution of the variational equality (I-7) for $q > 0$ (for a given $B > 0$). Then

i) The mappings

$$(1) \qquad q > 0 \to u_q \in V \qquad \text{and} \qquad q > 0 \to \int_{\Gamma_2} u_q \, d\gamma \in \mathbb{R}$$

are strictly decreasing functions.

ii) For all $q > 0$ and $h > 0$ we have the following estimates :

$$(2) \qquad \| \tfrac{1}{h} (u_{q+h} - u_q) \|_V \leq C_1 = \frac{\| \gamma_0 \|}{\alpha_0} \, | \Gamma_2 |^{1/2} ,$$

$$(3) \qquad \| \tfrac{1}{h} (u_q - u_{q+h}) \|_{L^2(\Gamma_2)} \leq C_2 = C_1 \, \| \gamma_0 \| ,$$

where γ_0 is the trace operator (linear and continuous, defined on V), and $\alpha_0 > 0$ is the coercivity constant on V_0 of the bilinear a , i.e. :

$$(4) \qquad a(v,v) \geq \alpha_0 \, \| v \|_V^2 , \; \forall v \in V_0 .$$

iii) For all $q > 0$ and $h > 0$ we have

$$(5) \qquad 0 < \int_{\Gamma_2} u_q \, d\gamma - \int_{\Gamma_2} u_{q+h} \, d\gamma \leq C_3 \, h \quad (C_3 = C_2 \, | \Gamma_2 |^{1/2} > 0)$$

and therefore the function $q > 0 \to \int_{\Gamma_2} u_q \, d\gamma$ is continuous.

Proof. If $u_i = u_{q_i}$ is the solution of (I-7) for $q_i > 0$ ($i = 1, 2$), then we have the following equalities :

$$(6) \qquad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1) \, d\gamma ,$$

$$(7) \qquad a(u_2,u_2) - a(u_1,u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2) \, d\gamma ,$$

because we take $v = u_2 \in K$ in the variational equality corresponding to u_1 , and $v = u_1 \in K$ in the one corresponding to u_2 , and we add up and subtract both equalities. From (6) and (7) we obtain (2) and (3).

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be the real function defined by

$$(8) \qquad f(q) = J(u_q) = \tfrac{1}{2} \, a(u_q,u_q) + q \int_{\Gamma_2} u_q \, d\gamma .$$

Remark 2. To solve Problem 1 it is sufficient to find a value $q > 0$ for which we have $f(q) < 0$. We shall further see that this technique can still be improved.

THEOREM 2. i) The function f is differentiable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

$$f'(q) = \int_{\Gamma_2} u_q \, d\gamma . \tag{9}$$

ii) There exists a constant $C > 0$ such that

$$a(u_q,u_q) = C\, q^2 , \tag{10}$$

$$f(q) = -\frac{C}{2} q^2 + B \mid \Gamma_2 \mid q . \tag{11}$$

iii) If $q > q_0(B)$, then we obtain a two-phase steady-state Stefan problem in Ω (i.e. u_q is a function of non-constant sign in Ω), where

$$q_0(B) = \frac{B \mid \Gamma_2 \mid}{C} . \tag{12}$$

iv) Constant $C = C(\Omega,\Gamma_1,\Gamma_2) > 0$ is given by

$$C = a(u_3,u_3) = \int_{\Gamma_2} u_3 \, d\gamma , \tag{13}$$

where u_3 is the solution of the variational equality

$$a(u_3,v) = \int_{\Gamma_2} v \, d\gamma , \quad \forall v \in V_0 , \quad u_3 \in V_0 . \tag{14}$$

Moreover, C can be calculated by

$$C = \frac{1}{q} \int_{\Gamma_2} (B - u_q) \, d\gamma , \tag{15}$$

for some $q > 0$.

Proof. We deduce (9) by considering the fact that

$$\frac{f(q + h) - f(q)}{h} = \frac{1}{2} \int_{\Gamma_2} u_q \, d\gamma + \frac{1}{2} \int_{\Gamma_2} u_{q+h} \, d\gamma . \tag{16}$$

which is obtained from (I-7) after elementary manipulations.

Moreover, we have

(17) $u_q = B - q\, u_3 \quad \text{in } \Omega$,

(18) $f'(q_0(B)) = 0$.

We obtain the thesis by using the fact that if $\int_{\Gamma_2} u_q \, d\gamma < 0$ then $u_q^- \neq 0$ in $\bar{\Omega}$.

Remark 3. The sufficient condition $f(q) < 0$, to solve Problem 1, was improved by the condition $f'(q) < 0$, which is optimal (see examples more later). In the case where, because of symmetry, we find that the function u_q is constant on Γ_2 , the sufficient condition, given by (Th.2-iii), is also necessary to have a steady-state two-phase Stefan problem.

Remark 4. Constant C has the physical dimension :

(19) $[C] = (cm)^n$

where n is the dimension of the space $\mathbb{R}^n$.

COROLLARY 3. If we consider the general case $b = b(x) > 0$ on Γ_2 , we obtain : If function q satisfies the inequality

(20) $$\operatorname*{Inf}_{x \in \Gamma_2} q(x) > \frac{k_2 \mid \Gamma_2 \mid}{C} \operatorname*{Sup}_{x \in \Gamma_1} b(x)$$

then we have a two-phase steady-state Stefan problem in Ω, that is function $u = u_{qb}$ is of non-constant sign in Ω.

Proof. We apply part (iii) of Theorem 2 and the monotony property (I-11).

Let $q_c > 0$ be the critical heat outgoing flux which characterizes a steady-state two-phase Stefan problem, that is

(21)
$$q > q_c \Leftrightarrow \exists \text{ 2-phases,}$$
$$q \leq q_c \Leftrightarrow \exists \text{ 1-phase (the liquid phase).}$$

We shall give now some estimates for the critical flux q_c [BST].

LEMMA 4. i) Let w denote the solution to

(22) $$\Delta w = 0 \text{ in } \Omega \,, \quad w \mid_{\Gamma_1} = B \,, \quad w \mid_{\Gamma_2} = 0 \,, \quad \frac{\partial w}{\partial n} \mid_{\Gamma_3} = 0 \,.$$

If we define

(23) $$q_i = \operatorname{Min}_{\Gamma_2} \left(- \frac{\partial w}{\partial n} \mid \Gamma_2\right)$$

then $u_q \geq w \geq 0$ in $\overline{\Omega}$, $\forall q \leq q_i$. Moreover, we have

(24) $$q_i \leq q_c \ .$$

ii) Let $P_2 \in \Gamma_2$ and the affine function π such that

(25) $$\begin{aligned} &\pi \mid_{\Gamma_1} \geq B \ , \\ &\pi(P_2) = 0 \ , \ \pi \mid_{\Gamma_2} \geq 0 \ , \\ &\frac{\partial \pi}{\partial n} \mid \Gamma_3 \geq 0 \ . \end{aligned}$$

If we define

(26) $$q_s = \operatorname{Max}_{\Gamma_2} \left(- \frac{\partial \pi}{\partial n} \mid \Gamma_2\right)$$

then $u_q \leq \pi$ in Ω, $\forall q \geq q_s$. Moreover, we have $u_q(P_2) < 0$, $\forall q > q_s$ and then

(27) $$q_c \leq q_s \ .$$

iii) On the other hand $w \leq \pi$ in $\overline{\Omega}$ and if $w \neq \pi$ we have $q_i < q_s$.

Proof. We apply the maximum principle [KS, PW].

Remark 5. A sufficient condition for such π to exist is the existence of supporting hyperplanes σ to Ω at $P_2 \in \Gamma_2$ which are a positive distance away from $\overline{\Gamma}_1$: construct an affine function π vanishing on σ (and at P_2), such that $\pi \mid_{\Gamma_1} \geq B$ and there is $P_1 \in \overline{\Gamma}_1$ with $\pi(P_1) = B$. The optimal q_s can be obtained by selecting P_2 , $\sigma = \sigma_{P_2}$ such that $\operatorname{dist}(\sigma, \overline{\Gamma}_1)$ is the largest. This construction fails if Γ_2 is a flat portion of Γ, e.g. the side of a triangle $\Omega \subset \mathbf{R}^2$, Γ_1 being formed by the other two sides and $\Gamma_3 = \emptyset$. The fact that $u_q(P_2) < 0$ suggests that the second phase appears at $P_2 \in \Gamma_2$, the point "farthest" from Γ_1 . In many cases (c.f. [BST]) the function π can be obtained by satisfying (25) and $\pi(P_1) = B$, where $P_1 \in \overline{\Gamma}_1$, $P_2 \in \Gamma_2$ and $\operatorname{dist}(P_1, P_2) = \sup_{x \in \Gamma_2} \operatorname{dist}(x, \overline{\Gamma}_1)$. There is no uniqueness in general for the points $P_1 \in \overline{\Gamma}_1$ and $P_2 \in \Gamma_2$. For instance, in Example 1 (see below) there are many $P_1 = (0,y)$ and $P_2 = (x_0,y)$, with $y \in [0,y_0]$.

We shall consider $q_c = q_c(\Omega)$ as a function of the domain Ω. Let Ω_1 and Ω_2 be two bounded domains, with regular boundaries, such that [BST] :

(28) $$\Omega_1 \subset \Omega_2 \ , \ \partial(\Omega_1) = \Gamma_1^{(1)} \cup \Gamma_2 \cup \Gamma_3 \ , \ \ \partial(\Omega_2) = \Gamma_1^{(2)} \cup \Gamma_2 \cup \Gamma_3 \ ,$$

where the boundary conditions on $\Gamma_1^{(i)}$ (i = 1, 2), Γ_2 and Γ_3 are of the same type as the ones defined before. Let u_i (i = 1, 2) be the solution to problem (I-7) for the domain Ω_i with data $B = B(x)>0$ on $\Gamma_1^{(i)}$ and $q_i = q_i(x)$ on Γ_2 (i = 1, 2), that is

$$(29) \qquad a_i(u_i, v-u_i) = -\int_{\Gamma_2} q_i\,(v-u_i)\,d\gamma \ , \ \forall v \in K_i \ , \ u_i \in K_i \qquad (i = 1, 2) \ ,$$

where

$$(30) \qquad K_i = \left\{ v \in H^1(\Omega_i) \,/\, v\,|_{\Gamma_1^{(i)}} = B \right\} , \ a_i(u,v) = \int_{\Omega_i} \nabla u \,.\, \nabla v \,dx \ \ (i = 1, 2) \,.$$

THEOREM 5. Under the above hypotheses, we obtain the following property :

$$(31) \qquad q_1 \le q_2 \ \text{ on } \Gamma_2 \ \Rightarrow \ u_2 \le u_1 \ \text{ in } \bar{\Omega}_1 \,.$$

Moreover, we have that

$$(32) \qquad q_C(\Omega_2) \le q_C(\Omega_1) \ ,$$

that is, $q_C = q_C(\Omega)$ is a non-increasing function of the domain Ω where the order is represented by conditions (28).

Proof. To prove (31) we shall take into account the following equivalence

$$(33) \qquad u_2 \le u_1 \text{ in } \bar{\Omega}_1 \ \Leftrightarrow \ z = 0 \text{ in } \bar{\Omega}_1 \ ,$$

where $z = (u_2 - u_1)^+ \in H^1(\Omega_1)$. Moreover, $z\,|_{\Gamma_1^{(1)}} = 0$ because $u_2 \le B$ in $\bar{\Omega}_2$, i.e. $u_2\,|_{\Gamma_1^{(1)}} \le B$. By using (29) with $v = u_1 + z \in K_1$, we have

$$(34) \qquad a_1(u_1,z) = -\int_{\Gamma_2} q_1\, z\, d\gamma \,.$$

If we extend by 0 the function z to the whole set Ω_2 and we put $v = u_2 + z \in K_2$ in (29), we obtain

$$(35) \qquad -\int_{\Gamma_2} q_2\, z\, d\gamma = a_2(u_2,z) = \int_{\Omega_2} \nabla u_2 \,.\, \nabla z \,dx = \int_{\Omega_1} \nabla u_2 \,.\, \nabla z\, dx = a_1(u_2,z) \ .$$

From (34) and (35) we obtain

$$0 \le a_1(z,z) = a_1(u_2 - u_1,z) = \int_{\Gamma_2} (q_1 - q_2)\, z\, d\gamma \le 0 \ ,$$

that is $z = 0$ in $\bar{\Omega}_1$. On the other hand, (32) follows from (31) by putting $q_1 = q_2\ (= q)$.

We shall give now another estimate for q_C, by using Poincaré type barriers. Let $\xi \in \Gamma_2$ be such that there exists $x_0 \notin \bar{\Omega}$, with

(36) $$\| x_0 - \xi \| = a > 0 \, , \; \left\{ x \, / \, \| x - x_0 \| \leq a \right\} \cap \bar{\Omega} = \left\{ \xi \right\} ,$$

where a is a positive parameter and $\| \cdot \|$ is the euclidean norm in $\mathbf{R}^n$. The Poincaré barriers at $\xi \in \Gamma_2$ are [Ke] :

(37) $$V_{D,a}(x,\xi) = V(x,\xi) = \begin{cases} D \left\{ \dfrac{1}{a^{n-2}} - \dfrac{1}{\| x - x_0 \|^{n-2}} \right\} , & (n \geq 3) \\ D \, \log \left(\dfrac{\| x - x_0 \|}{a} \right) , & (n = 2) \end{cases}$$

where D is a another positive parameter. Let $P_1 \in \bar{\Gamma}_1$, $P_2 \in \Gamma_2$ be such that

(38) $$d = \sup_{x \in \Gamma_2} \operatorname{dist}(x, \bar{\Gamma}_1) = \| P_2 - P_1 \| > 0 \; .$$

Let $\xi = P_2 \in \Gamma_2$ be . Then we have

THEOREM 6. We assume the following hypotesis :

(39) $$V \mid_{\Gamma_1} \geq B \iff V(P_1,\xi) \geq B \; .$$

Let q_V be defined by

(40) $$q_V = \inf_{V(P_1,\xi) = B} \left(\frac{D}{a^{n-1}} \right) .$$

Then we obtain that

(41) $$V(x,\xi) > u_q(x) \; , \; \forall x \in \Omega \; , \; \forall q > q_V$$

(42) $$q_C < q_V \; .$$

Proof. It following from the monotony property (I-11) and relations ([BST])

(43) $$\begin{aligned} &\Delta_x V(x,\xi) = 0 \quad \text{in } \Omega \, , \\ &\frac{\partial V}{\partial n}(x,\xi) = D \frac{(x - x_0) \, . \, n(x)}{\| x - x_0 \|^n} \; , \quad \forall x \in \Gamma_2 \, , \\ &\inf_{x \in \Gamma_2} \frac{\partial V}{\partial n}(x,\xi) = \frac{\partial V}{\partial n}(\xi,\xi) = - \frac{D}{a^{n-1}} \, , \\ &u_q(\xi) < 0 \; . \end{aligned}$$

Remark 6. Equivalence (39) is an immediate consequence of the monotonicity of V on $\| x - x_0 \|$, for special domains.

Remark 7. Let

(44) $$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \,/\, -E \leq x \leq E \,,\, -h \leq y \leq h \right\}, E > 0 \;,\; h > 0 \;.$$

Let Γ_1 be the top and bottom sides of this rectangle, and let Γ_2 be the two vertical sides. We maintain a temperature $b > 0$ on Γ_1 ($B = k_2 b > 0$) and ask for the minimum heat flux q on Γ_2 for which the zone $\left\{ (x,y) \in \Omega \,/\, u(x,y) > 0 \right\}$ (whose boundary obviously contains Γ_1) is disconnected, a region where $u < 0$ joins the two components of Γ_2 . By introducing a variant of the Poincaré barriers (37) we obtain that [BST]

(45) $$q > \frac{2\,e\,B\,E}{h^2 - E^2} \;\text{(with } h > E) \;\Rightarrow\; \left\{ u > 0 \right\} \text{ is disconected.}$$

Problem 2 : For the general case $b = b(x) > 0$ on Γ_1 and $q = q(x)$ on Γ_2 , we consider the following optimization problem : Find $q \in Q^+$ that produces the maximum heat flux on Γ_2 , without change of phase within Ω , i.e. [GT1] :

(46) $$\underset{q \in Q^+}{\text{Max}} \; F(q)$$

where

$$F : Q \rightarrow \mathbb{R} \;/\; F(q) = \int_{\Gamma_2} q \, d\gamma \,,$$

(47) $$Q = H^{1/2}(\Gamma_2) \;,\; S = \left\{ v \in K \,/\, \Delta v = 0 \text{ in } \Omega \,,\, \frac{\partial v}{\partial n} \Big|_{\Gamma_3} = 0 \right\},$$

$$S^+ = \left\{ v \in S \,/\, v \geq 0 \text{ in } \Omega \right\} \;,\; Q^+ = T^{-1}(S^+) = \left\{ q \in Q \,/\, u_q \geq 0 \text{ in } \Omega \right\} .$$

The application $T : Q \rightarrow S$ is defined by

(48) $$T(q) = u$$

where $u = u_q$ is the unique solution of (I-7). We consider that the domain Ω and the data B on Γ_1 (e.g. $B \in H^{3/2}(\Gamma_1)$) and q on Γ_2 (e.g. $q \in Q$) are sufficiently regular to have the regularity property $u \in H^2(\Omega) \cap C^0(\bar{\Omega})$ (for $n \leq 3$, $H^2(\Omega) \subset C^0(\bar{\Omega})$) [Fri, Gr, MS, Ne]. Moreover, in the three examples given below, the solution satisfies this condition for the constant case. Therefore, we have that there will not exist a phase change in Ω for any heat flux $q \in Q^+$.

THEOREM 7. (i) The operator T, defined by (48), is an affine and monotone increasing operator, that is, there exist $u_1 \in S$ and two new operator T_1 and T_2 so that $T = T_1 + T_2$, where

(49)
$$T_1 : Q \rightarrow S \;/\; T_1(q) = u_1 \in S \;,\; \forall q \in Q \;,$$
$$T_2 : Q \rightarrow V_0 \;/\; T_2 \text{ is linear and continuous.}$$

(ii) Q^+ is a convex set and F is a linear (then, convex) functional.

(iii) There exists a unique $\overline{q} \in Q^+$ such that

(50)
$$F(\overline{q}) = \underset{q \in Q^+}{\text{Max}} F(q) \;.$$

Moreover, the element $\overline{q}$ is defined by

(51)
$$\overline{q} = - \frac{\partial \omega}{\partial n} \Big|_{\Gamma_2}$$

where ω is given by (22).

Proof. (i) Let $u_1 \in K$ and $u_2 = u_2(q) \in V_0$ be the unique solutions to the following problems [KS] :

(52)
$$a(u_1, v - u_1) = 0 \;,\; \forall v \in K \,,\; u_1 \in K \,,$$

(53)
$$a(u_2, v) = - \int_{\Gamma_2} q\, v\, d\gamma \;,\quad \forall v \in V_0 \,,\, u_2 \in V_0 \;.$$

We have that $u = u_1 + u_2$ from the uniqueness of problem (I-7); then T_1 and T_2 can be defined as follows :

(54)
$$T_1(q) = u_1 \;,\; T_2(q) = u_2 \;,\; \forall q \in Q \;.$$

(ii) Q^+ is a convex set due to the fact that T is an affine operator and S^+ is a convex set.

(iii) The element $\overline{q} \in Q^+$ verifies $F(q) \leq F(\overline{q})$, $\forall q \in Q^+$, by using the maximum principle.

Let $I : S \rightarrow \mathbf{R}$ be the linear functional, defined by :

(55)
$$I(v) = - \int_{\Gamma_2} \frac{\partial v}{\partial n}\, d\gamma \;,\; \forall v \in S \,.$$

We can consider a new formulation of the optimization problem (46), as follows :

(56)
$$\underset{v \in S^+}{\text{Max}} I(v) \;.$$

Let Ψ, P and G_0 be

(57)
$$\Psi = C^0(\Gamma_2) \ , \ P = \left\{ p \in \Psi \ / \ p \geq 0 \text{ on } \Gamma_2 \right\} \ \text{(cone)}$$
$$G_0 : S \rightarrow \Psi \ / \ G_0(v) = - v \mid_{\Gamma_2} \ ,$$

then the problem (56) is equivalent to

(58)
$$\underset{v \in S,\ G_0(v) \leq 0}{\text{Max}} \ I(v) \ .$$

If u is a solution of (58), there exists a Lagrange multiplier $\mu \in \Psi^*$ (dual of Ψ) with $\mu \geq 0$ (i.e. $<\mu,p> = \int_{\Gamma_2} \mu \, p \, d\gamma \geq 0 \ , \ \forall p \in P$) that satisfies the following conditions [Ben, ET] :

(59)
$$I(u) + <\mu, G_0(v)> \geq I(v) \ , \ \ \forall v \in S \ ,$$
$$<\mu, G_0(u)> = 0 \ .$$

We can deduce that $u = \omega$ and $\mu = \frac{\partial Z_0}{\partial n} \mid_{\Gamma_2}$, where the element Z_0 is given by

(60)
$$\Delta Z_0 = 0 \ \text{ in } \Omega \ , \ Z_0 \mid_{\Gamma_1} = 0 \ , \ Z_0 \mid_{\Gamma_2} = 1 \ , \ \frac{\partial Z_0}{\partial n} \mid_{\Gamma_3} = 0 \ ,$$

and therefore we obtain the uniqueness of the solution of (50).

Problem 3 : For the general case $b = b(x) > 0$ on Γ_1 and $q = q(x) > 0$ on Γ_2 , we consider the following optimization problem : Find the maximum upper bound for q such that there is no change of phase within Ω, i.e. [GT1]

(61)
$$\text{Find } q_M^0 > 0 \ / \ u_q \geq 0 \text{ in } \Omega \ , \ \forall q = q(x) \leq q_M^0 \text{ on } \Gamma_2 \ .$$

THEOREM 8. (i) For the case $q = \text{const.} > 0$, we obtain that

(62)
$$q_M^0 = \underset{x \in \Gamma_2}{\text{Inf}} \ \frac{u_1(x)}{u_2(x)} \ ,$$

where u_1 and u_3 are given respectively by

(63)
$$\Delta u_1 = 0 \ \text{ in } \Omega \ , \ u_1 \mid_{\Gamma_1} = B \ , \ \frac{\partial u_1}{\partial n} \mid_{\Gamma_2 \cup \Gamma_3} = 0 \quad \text{(c.f. (52))},$$

(64)
$$\Delta u_3 = 0 \ \text{ in } \Omega \ , \ u_3 \mid_{\Gamma_1} = 0 \ , \ \frac{\partial u_3}{\partial n} \mid_{\Gamma_2} = 1 \ , \ \frac{\partial u_3}{\partial n} \mid_{\Gamma_3} = 0 \quad \text{(c.f. (14))}.$$

(ii) If $q = q(x) > 0$ on Γ_2 satisfies the condition

(65) $$\sup_{x \in \Gamma_2} q(x) \le q_M^0 ,$$

where q_M^0 is defined by (62), then $u_q \ge 0$ in Ω.

(iii) For the constant case, we have that

(66) $$q_M^0 = q_c ,$$

where q_c is the critical heat outgoing flux (21).

Proof. (i) follows from Theorem 7 with $u_2(q) = -q\,u_3$, (ii) from Lemma 1 and (iii) from the definition of q_M^0 and q_c respectively.

Problem 4. For the constant case $B > 0$, $q > 0$ and $\alpha > 0$, find conditions between α, q (for a given $B > 0$) to have a steady-state two-phase Stefan problem in Ω, that is the solution u of (I-21) is a function of non-constant sign in Ω.

We shall consider that the domain Ω and the data b (or B) on Γ_1 and q on Γ_2 are sufficiently regular to have the regularity property $u_{\alpha qB} \in H^2(\Omega) \cap C^0(\overline{\Omega})$. Moreover, in the three examples, the solution $u_{\alpha qB}$ satisfies this requirement.

Remark 8. (i) The problem (I-21) is a two-phase Stefan problem in Ω if and only if :

(67) $$\exists\, x_1 \in \Gamma_1 , x_2 \in \Gamma_2 \;/\; u_{\alpha qB}(x_1) > 0 \;,\; u_{\alpha qB}(x_2) < 0 \;.$$

(ii) If $u_{\alpha qB}$ satisfies the following condition

(68) $$\int_{\Gamma_1} u_{\alpha qB}\, d\gamma > 0 \;,\; \int_{\Gamma_2} u_{\alpha qB}\, d\gamma < 0$$

then the problem (I-21) is a two-phase problem.

LEMMA 9. For all $B > 0$, we have the following expressions :

(69) $$\int_{\Gamma_1} u_{\alpha qB}\, d\gamma = B\,|\,\Gamma_1\,| - \frac{q}{\alpha}\,|\,\Gamma_2\,| \;,\; \forall \alpha, q > 0 ,$$

(70) $$a(u_{qB}, u_{qB}) = L_q(u_{qB}) + B\,q\,|\,\Gamma_2\,| \;,\; \forall q > 0 ,$$

(71) $$a(u_{\alpha qB}, u_{qB}) = a(u_{qB}, u_{qB}) \;,\; \forall \alpha, q > 0 .$$

Proof. It is enough to choose $v = 1 \in V$ in (I-21) and to use the definition of u_{qB} and $u_{\alpha qB}$, given by (I-7) and (I-21) respectively.

THEOREM 10. If $q > q_0(B)$, then (I-21) is a steady-state two-phase Stefan problem in Ω for all $\alpha > \alpha_0(q,B)$, where

$$(72) \qquad \alpha_0(q,B) = \frac{q\,|\Gamma_2|}{B\,|\Gamma_1|} .$$

Proof. As $q > q_0(B)$, we have that

$$(73) \qquad \underset{\Omega}{\mathrm{Min}}\, u_{qB} = \underset{\Gamma_2}{\mathrm{Min}}\, u_{qB} < 0 ,$$

and therefore, by using (I-25-ii), we deduce that $M_2 < 0$, $\forall \alpha > 0$. Besides, by using (69), we have that

$$(74) \qquad \int_{\Gamma_1} u_{\alpha qB}\, d\gamma > 0 \iff \alpha > \alpha_0(q,B),$$

then we obtain the thesis.

Remark 9. In the case where, due to symmetry, we find that function $u_{\alpha qB}$ is constant on Γ_1, then the sufficient condition, given by Theorem 10, is also necessary for problem (I-21) to be a steady-state two-phase Stefan problem.

Let $g : (\mathbf{R}^+)^3 \to \mathbf{R}$ be the real function defined by

$$(75) \qquad g(\alpha,q,B) = G_{\alpha qB}(u_{\alpha qB}) , \quad \alpha, q, B > 0 ,$$

which is equivalent to the following expressions

$$(76) \qquad g(\alpha,q,B) = -\tfrac{1}{2}\, a_\alpha(u_{\alpha qB}, u_{\alpha qB}) = -\tfrac{1}{2}\, L_{\alpha qB}(u_{\alpha qB}) = \frac{q}{2}\int_{\Gamma_2} u_{\alpha qB}\, d\gamma - \frac{\alpha B}{2}\int_{\Gamma_1} u_{\alpha qB}\, d\gamma \le 0 .$$

THEOREM 11. (i) Function g has partial derivatives with respect to variables α, q and B, and they are given by the following expressions for all α, q, $B > 0$:

$$(77) \qquad \frac{\partial g}{\partial \alpha}(\alpha,q,B) = \int_{\Gamma_1} \left(\tfrac{1}{2} u^2_{\alpha qB} - B\, u_{\alpha qB} \right) d\gamma ,$$

(78) $$\frac{\partial g}{\partial q}(\alpha,q,B) = \int_{\Gamma_2} u_{\alpha qB}\, d\gamma \ ,$$

(79) $$\frac{\partial g}{\partial B}(\alpha,q,B) = -\alpha \int_{\Gamma_1} u_{\alpha qB}\, d\gamma \ .$$

(ii) There exists a function $A = A(\alpha) > 0$, defined for $\alpha > 0$, such that

(80) $$g(\alpha,q,B) = -\frac{A(\alpha)}{2} q^2 + B\, q \mid \Gamma_2 \mid - \frac{B^2 \alpha}{2} \mid \Gamma_1 \mid ,$$

(81) $$\int_{\Gamma_2} u_{\alpha qB}\, d\gamma = B \mid \Gamma_2 \mid - q\, A(\alpha) \ , \quad \forall q,\ B > 0 \, .$$

(iii) Function $A = A(\alpha)$ is a decreasing positive function of α which verifies

$$A(\alpha) > \frac{\mid \Gamma_2 \mid^2}{\mid \Gamma_1 \mid} \frac{1}{\alpha} \quad (\text{ then } A(0^+) = +\infty)$$

(82) $$\lim_{\alpha \to +\infty} A(\alpha) = C \ , \qquad \lim_{\alpha \to +\infty} \alpha A'(\alpha) = 0 \, ,$$

$$(\alpha A(\alpha))' = \frac{1}{q^2}\, a(u_{\alpha qB}, u_{\alpha qB}) \ ,$$

where $C > 0$ is the constant defined in Theorem 2.

Proof. (i) For example, for the partial derivative of g with respect to α, we can obtain :

(83) $$\| \Delta_h u(\alpha) \|_V \leq E_1 h \ , \ \| \Delta_h u(\alpha) \|_{L^2(\Gamma_1)} \leq E_2 h \ ,$$

where

(84) $$E_1 = \frac{\| \gamma_0 \|}{\lambda_\alpha} \| B - u_{\alpha qB} \|^2_{L^2(\Gamma_1)} \ , \ E_2 = E_1 \| \gamma_0 \| \ ,$$

and $\Delta_h u(\alpha)$ is defined by ($h = \text{const.} > 0$)

(85) $$\Delta_h u(\alpha) = u_{\alpha+h,qB} - u_{\alpha qB} \in V \ .$$

Moreover, we can obtain

(86) $$\frac{\Delta_h g(\alpha)}{h} = \frac{1}{2} \int_{\Gamma_1} \Big(u_{\alpha+h,qB}\, u_{\alpha qB} - B\,(u_{\alpha+h,qB} + u_{\alpha qB}) \Big)\, d\gamma \ ,$$
$$\lim_{h \to 0^+} \int_{\Gamma_1} u_{\alpha+h,qB}\, u_{\alpha qB}\, d\gamma = \int_{\Gamma_1} u^2_{\alpha qB}\, d\gamma \ ,$$

where $\Delta_h g(\alpha)$ is defined by

(87) $$\Delta_h g(\alpha) = g(\alpha+h,q,B) - g(\alpha,q,B) \ .$$

We can given an analogous definition for $\Delta_h u(q)$, $\Delta_h g(q)$ and $\Delta_h u(B)$, $\Delta_h g(B)$ [TT] .

(ii)−(iii) Function $u_{\alpha q B}$ can be expressed as follows

(88) $$u_{\alpha q B} = B - q\, U_\alpha \quad \text{in } \Omega ,$$

where U_α is defined by

(89) $$\Delta U_\alpha = 0 \ \text{in } \Omega , \quad -\frac{\partial U_\alpha}{\partial n} \Big|_{\Gamma_1} = \alpha\, U_\alpha , \quad \frac{\partial U_\alpha}{\partial n} \Big|_{\Gamma_2} = 1 , \quad \frac{\partial U_\alpha}{\partial n} \Big|_{\Gamma_3} = 0 ,$$

whose variational formulation is given by

(90) $$a_\alpha(U_\alpha, v) = \int_{\Gamma_2} v\, d\gamma , \quad \forall v \in V , \ U_\alpha \in V ,$$

and verifies that $U_\alpha > 0$ in $\bar{\Omega}$. Moreover, function U_α verifies the following properties :

(91) $$\int_{\Gamma_1} U_\alpha\, d\gamma = \frac{|\Gamma_2|}{\alpha} , \quad \forall \alpha > 0 ,$$

(92) $$\int_{\Gamma_2} U_\alpha\, d\gamma = A(\alpha) , \quad \forall \alpha > 0 ,$$

(93) $$a(U_\alpha, U_\alpha) = C , \quad \forall \alpha > 0 .$$

Taking into account [GT2, TT] the thesis is achieved.

We can complete the above results by the following Theorem [GT2,TT] :

THEOREM 12. (i) Problem (I-21) represents a steady-state two-phase Stefan problem if and only if the heat flux q verifies the following inequalities

(94) $$q_1(\alpha,B) < q < q_2(\alpha,B) , \quad \alpha > 0 , \ B > 0 ,$$

where $q_1 = q_1(\alpha,B)$ and $q_2 = q_2(\alpha,B)$ are given by

(95) $$q_1(\alpha,B) = \underset{\Gamma_2}{\text{Min}} \left(\frac{B}{U_\alpha}\right) , \quad q_2(\alpha,B) = \underset{\Gamma_1}{\text{Max}} \left(\frac{B}{U_\alpha}\right) .$$

(ii) Let $q_m = q_m(\alpha,B)$ and $q_M = q_M(\alpha,B)$ be real functions, defined for α, $B > 0$ by the following expressions

(96) $$q_m(\alpha,B) = \frac{B\,|\Gamma_2|}{A(\alpha)} , \quad q_M(\alpha,B) = \frac{B\,\alpha\,|\Gamma_1|}{|\Gamma_2|} .$$

They verifies the conditions

$$
(97)\qquad
\begin{cases}
q_m(0^+,B) = q_M(0^+,B) = 0 , \\
q_m(\alpha,B) < q_M(\alpha,B) , \ \forall \alpha > 0, B > 0 , \\
\lim_{\alpha \to +\infty} q_m(\alpha,B) = q_0(B) \quad \text{(defined by (II-12))} , \\
q_m \text{ is an increasing function of variable } \alpha ,
\end{cases}
$$

and the estimates

$$
(98)\qquad q_1(\alpha,B) \le q_m(\alpha,B) < q_M(\alpha,B) \le q_2(\alpha,B) , \ \forall \alpha, B > 0 .
$$

Moreover, we have

$$
(99)\qquad
\begin{aligned}
q_1(\alpha,B) = q_m(\alpha,B) &\Leftrightarrow U_\alpha \mid_{\Gamma_2} = \text{const.} \left(= \frac{A(\alpha)}{|\Gamma_2|} \right) , \\
q_2(\alpha,B) = q_M(\alpha,B) &\Leftrightarrow U_\alpha \mid_{\Gamma_1} = \text{const.} \left(= \frac{|\Gamma_2|}{\alpha |\Gamma_1|} \right) ,
\end{aligned}
$$

and the set

$$
(100)\qquad S^{(2)}(B) = \left\{ (\alpha,q) \in (R^+)^2 \,/\, q_m(\alpha,B) < q < q_M(\alpha,B) , \alpha > 0 \right\}
$$

is not empty, for all $B > 0$.

(iii) We have the following equivalences :

$$
(101)\qquad \text{i)} \int_{\Gamma_1} u_{\alpha q B}\, d\gamma > 0 \Leftrightarrow q < q_M(\alpha,B) , \quad \text{ii)} \int_{\Gamma_2} u_{\alpha q B}\, d\gamma < 0 \Leftrightarrow q > q_m(\alpha,B) .
$$

COROLLARY 13. If $(\alpha,q) \in S^{(2)}(B)$, then (I-21) is a two-phase steady-state Stefan problem.

Remark 9. In [Ta5] and [ST], the numerical results to compute $q_0(B)$ and the set $S^{(2)}(B)$ respectively were obtained by using the software MODULEF [Ber] (finite element code).

Remark 10. In the case where, due to symmetry, we find that $u_{\alpha q B}$ or U_α is constant on Γ_1 and Γ_2 respectively, then the sufficient condition, given by corollary 13 is also necessary for problem (I-21) to be a two-phase Stefan problem.

The function $A = A(\alpha)$, defined for $\alpha > 0$, is not explicitly known but has properties (82) and (92). Now, we shall consider a particular case for which we can obtain more information about the expression of $A(\alpha)$.

We consider the particular case when $u_{\alpha qB}$ verifies the condition [TT]

(102) $$\frac{1}{q^2}\, a(u_{\alpha qB}, u_{\alpha qB}) = \text{Const.} \; (= \text{Const}(\alpha,q,B)\,)\;,\; \forall \alpha,\, q,\, B > 0\;,$$

or in an equivalent way

(103) $$(\,\alpha\, A(\alpha)\,)' = A(\alpha) + \alpha\, A'(\alpha) = \text{const.}\;,\; \forall \alpha > 0\;,$$

due to (82). In this case, we have necessarily that

(104) $$\text{Const}(\alpha,q,B) = C > 0\;,\; \forall \alpha,\, q,\, B > 0\;,$$

where C is the constant defined in Theorem 2.

LEMMA 14. (i) We have the following equivalence

(105) $$u_{qB} - u_{\alpha qB} \text{ is constant in } \Omega \;\Leftrightarrow\; (\,\alpha\, A(\alpha)\,)' = C\,.$$

(ii) For the particular case (102), we have the following properties :

(106) $$u_{qB} - u_{\alpha qB} = \frac{q\,|\,\Gamma_2\,|}{\alpha\,|\,\Gamma_1\,|} \text{ in } \Omega\;,$$

(107) $$u_{\alpha qB}\,|_{\Gamma_1} = B - \frac{q\,|\,\Gamma_2\,|}{\alpha\,|\,\Gamma_1\,|}\;,$$

(108) $$\frac{\partial u_{\alpha qB}}{\partial n}\,|_{\Gamma_1} = \frac{q\,|\,\Gamma_2\,|}{|\,\Gamma_1\,|}\;,$$

(109) $$\frac{\partial u_{qB}}{\partial n}\,|_{\Gamma_1} = \text{const.}$$

Moreover, the function $A(\alpha)$ is given by the expression

(110) $$A(\alpha) = C + \frac{1}{\alpha}\,\frac{|\,\Gamma_2\,|^2}{|\,\Gamma_1\,|}\;.$$

Proof. (i) Owing to (71), we deduce

$$u_{qB} - u_{\alpha qB} \text{ is constant in } \Omega \;\Leftrightarrow\; a(u_{qB} - u_{\alpha qB}, u_{qB} - u_{\alpha qB}) = 0 \;\Leftrightarrow$$

$$\Leftrightarrow\; a(u_{\alpha qB}, u_{\alpha qB}) = a(u_{qB}, u_{qB}) \;\Leftrightarrow\; a(u_{\alpha qB}, u_{\alpha qB} = C\,q^2 \;\Leftrightarrow\; (\,\alpha\, A(\alpha)\,)' = C\;.$$

Remark 11. For the particular case (102), a complete description of the set $S^{(2)}(B)$ was obteined.

We shall give three examples in which the solution is explicitly known [Ta2] so that we can verify all the theoretical results obtained in this work.

Example 1. We consider the following data

$$n = 2 \ , \ \Omega = (0,x_0) \times (0,y_0) \ , \ x_0 > 0 \ , \ y_0 > 0 \ ,$$

$$\Gamma_1 = \{0\} \times [0,y_0] \ , \ \Gamma_2 = \{x_0\} \times [0,y_0] \ , \tag{111}$$

$$\Gamma_3 = (0,x_0) \times \{0\} \cup (0,x_0) \times \{y_0\}$$

Example 2. Next we consider

$$n = 2 \ , \ 0 < r_1 < r_2 \ , \ \Gamma_3 = \phi \ ,$$

Ω : annulus of radius r_1 and r_2, centered at (0,0) ,

(112) Γ_1 : circumference of radius r_1 and center (0,0) ,

Γ_2 : circumference of radius r_2 and center (0,0) .

Example 3. Finally, we take into account the same information of Example 2 but now for the case $n = 3$.

Remark 12. The three examples verifies condition (102), that is, they are particular cases.

REFERENCES

[Ba] C. BAIOCCHI, "Sur un problème à frontière libre traduisant le filtrage de liquides à travers des milieux poreux", C.R. Acad. Sc. Paris, 273A(1971), 1215-1217.

[BC] C. BAIOCCHI - A. CAPELO, "Disequazioni variazionali e quasivariazionali. Applicazioni a problemi di frontiera libera", Vol. 1: Problemi variazionali, Vol. 2: Problemi quasivariazionali, Quaderni dell'Unione Matematica Italiana, N 4, 7, Pitagora Editrice, Bologna (1978).

[Ben] A. BENSOUSSAN , "Teoría moderna de control óptimo ", CUADERNOS del Instituto de Matemática "Beppo Levi", No. 7, Rosario (1974).

[Ber] M. BERNADOU - P.L. GEORGE - A. HASSIM - P. JOLY - P. LAUG - A. PERRONNET - E. SALTEL - D. STEER - G. VANDERBORCK - M. VIDRASCU, "MODULEF : Une bibliothèque modulaire d'éléments finis", INRIA, Roquencourt (1985).

[BST] J.E. BOUILLET - M. SHILLOR - D.A. TARZIA, "Critical outflow for a steady-state Stefan problem", Applicable Analysis, 32(1989), 31-51.

[Ca] J.R. CANNON, "The one-dimensional heat equation", Addison-Wesley, Menlo Park, California (1984).

[CJ] H.S. CARSLAW - J.C. JAEGER, "Conduction of heat in solids", Clarendon Press, Oxford (1959).

[Cr] J. CRANK, "Free and moving boundary problems", Clarendon Press, Oxford (1984).

[Di] J.I. DIAZ, "Nonlinear partial differential equations and free boundaries", Vol. I : Elliptic equations, Research Notes in Math. N_0 106, Pitman, London (1985).

[Du1] G. DUVAUT, "Résolution d'un problème de Stefan (Fusion d'un bloc de glace à zéro degré)", C.R. Acad. Sc. Paris, 276A(1973), 1461-1463.

[Du2] G. DUVAUT, "Problèmes à frontière libre en théorie des milieux continus", Rapport de Recherche N^0 185, LABORIA - IRIA, Rocquencourt (1976).

[Du3] G. DUVAUT, "Méthodes variationnelles, initiation et applications", Univ. Pierre-et-Marie-Curie (Paris VI), Paris (1979).

[ET] I. EKELAND - R. TEMAM, "Analyse convexe et problèmes variationnelles", Dunod - Gauthier Villars, Paris (1973).

[EO] C.M. ELLIOTT - J.R. OCKENDON, "Weak and variational methods for moving boundary problems", Research Notes in Math., N_0 59, Pitman, London (1982).

[Fre] M. FREMOND, "Diffusion problems with free boundaries", in Autumn Course on

Applications of Analysis to Mechanics, ICTP, Trieste (1976).

[Fri] A. FRIEDMAN, "Variational principles and free boundary problems", J. Wiley, New York (1982).

[GT1] R.L.V. GONZALEZ - D.A. TARZIA, "Optimization of heat flux in domain with temperature constraints", J. Optimiz. Th. Appl., To appear.

[GT2] R.L.V. GONZALEZ - D.A. TARZIA, "On some thermic flux optimization problems in a domain with Fourier boundary condition and state restrictions", To appear.

[Gr] P. GRISVARD, "Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain", in Numerical Solution of Partial Differential Equations III, SYNSPADE 1975, B. Hubbard(Ed.). Academic Press, New York (1976), 207-274.

[Ke] O.D. KELLOGG, "Foundations of potential theory", Springer Verlag, Berlin (1929.

[KS] D. KINDERLEHRER – G. STAMPACCHIA, "An introduction to variational inequalities and their applications", Academic Press, New York (1980).

[Li] J.L. LIONS, "Problèmes aux limites dans les équations aux derivées partielles", Presses de l'Univ. de Montréal, Montréal (1962).

[Ma1] E. MAGENES, "Topics in parabolic equations : some typical free boundary problems", in Boundary Value Problems for Linear Evolution Partial Differential Equations, H.G. Garnier (Ed.), D. Reidel Publ. Comp., Dordrecht (1976), 239-312.

[Ma2] E. MAGENES, "Problemi di Stefan bifase in piu variabili spaziali", Le Matematiche, 36(1981), 65-108.

[MS] M.K.V. MURTHY - G. STAMPACCHIA, "A variational inequality with mixed boundary conditions", Israel J. Math., 13(1972), 188-224.

[Ne] J. NECAS, "Les méthodes directes en théorie des équations elliptiques", Masson, Paris (1967).

[Pr] M. PRIMICERIO, "Problemi di diffusione a frontiera libera", Bollettino Un. Mat. Italiana, 18A(1981), 11-68.

[PW] M. H. PROTTER - H. F. WEINBERGER, "Maximum principles in differential equations", Prentice-Hall, Englewood Cliffs, N. J. (1967).

[Ro] J.F. RODRIGUES, "Obstacle problems in mathematical physics", North-Holland Mathematics Studies N_0 134, North-Holland, Amsterdam (1987).

[Ru] L.I. RUBINSTEIN, "The Stefan problem", Translations of Mathematical Monographs, Vol.

27, Amer. Math. Soc., Providence, R.I. (1971).

[ST] M.S. SANZIEL - D.A. TARZIA, "Cálculo numérico de problemas elípticos mixtos con presencia de cambio de fase mediante el sofware científico Modulef", in Mecánica Computacional, Vol.8, MECOM'88, L.A. Godoy- F.Flores (Eds.), AMCA, Santa Fe (1989), 117-125.

[TT] E.D. TABACMAN - D.A. TARZIA, "Sufficient and/or necessary conditions for the heat transfer coefficient on Γ_1 and the heat flux on Γ_2 to obtain a steady-state two-phase Stefan problem", J. Diff. Eq., 77(1989), 16-37.

[Ta1] D.A. TARZIA, "Sur le problème de Stefan à deux phases", Thèse 3ème Cycle, Univ. Paris VI (Mars 1979). See also C.R. Acad. Sc. Paris, 288A(1979), 941-944; Math. Notae, 27(1979/80), 145-156 and 157-165.

[Ta2] D.A. TARZIA, "Sobre el caso estacionario del problema de Stefan a dos fases", Math. Notae, 28(1980/81), 73-89.

[Ta3] D.A. TARZIA, "Introducción a las inecuaciones variacionales elípticas y sus aplicaciones a problemas de frontera libre", Centro Latinoamericano de Matemática e Informática, CLAMI - CONICET, No. 5, Buenos Aires (1981).

[Ta4] D.A. TARZIA, "Una revisión sobre problemas de frontera móvil y libre para la ecuación del calor. El problema de Stefan", Math. Notae, 29(1981), 147-241. See also "A bibliography on moving-free boundary problems for the heat-diffusion equation. The Stefan problem", (with 2528 references), Progetto Nacionale M.P.I. "Equazioni di evoluzione e applicazioni fisico-matematiche", Firenze (1988).

[Ta5] D.A. TARZIA, "Una desigualdad para el flujo de calor constante a fin de obtener un problema estacionario de Stefan a dos fases", Mecánica Computacional, Vol. 2, S.R. Indelsohn (Ed.), EUDEBA, Santa Fe (1985), 359-370. See also "An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem", Engineering Analysis, 5(1988), 177-181.

[Ta6] D.A. TARZIA, "The two-phase Stefan problem and some related conduction problems", Reuniões em Matemática Aplicada e Computacão Científica, Vol. 5, SBMAC, Gramado (1987). See also "On heat flux in material with or without phase change", in Int. Colloqium on Free Boundary Problems : Theory and Aplications, Irsee/Bavaria, 11-20 June 1987, Research Notes in Math., Pitman, To appear.

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