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INVERSION OF ULTRAHYPERBOLIC BESSEL OPERATORS

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ABSTRACT. Let $G_\alpha = G_\alpha(P \pm io, m, n)$ be the causal (anticausal) distribution defined by

$$G_\alpha(P \pm oi,m,n) = H_\alpha(m,n)\ (P \pm io)^{\frac{1}{2}(\frac{\alpha-n}{2})}\ K_{\frac{n-\alpha}{2}}[m(P \pm io)^{\frac{1}{2}}] ,$$

where m is a positive real number, $\alpha \in C$, K_μ designates the modified Bessel function of the third kind and $H_\alpha(m,n)$ is the constant defined by

$$H_\alpha(m,n) = \frac{e^{\pm i\frac{\pi}{2}q}\ e^{i\frac{\pi}{2}\alpha}\ 2^{1-\frac{\alpha}{2}}\ (m^2)^{\frac{1}{2}(\frac{n-\alpha}{2})}}{(2\pi)^{\frac{n}{2}}\ \Gamma(\frac{\alpha}{2})} .$$

The distributions $G_{2k}(P \pm io,m,n)$, where n = integer $\geqslant 2$ and $k = 1,2,\ldots$, are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator, iterated k-times:

$$K^k\{G_{2k}\} = \delta ;$$

$$K = \left\{ \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_n^2} - m^2 \right\}^k .$$

Let $B^\alpha f$ be the ultrahyperbolic Bessel operator defined by the formula

$$B^\alpha f = G_\alpha * f ,$$

$f \in S$.

Our problem consists in the obtainment of an operator $T^\alpha = (B^\alpha)^{-1}$ such that if

$$B^\alpha f = \varphi ,$$

then

$$T^\alpha \varphi = f .$$

In this Note we prove (Theorem III.1, formula (III,7)) that

$$T^\alpha = G_{-\alpha} ,$$

for all $\alpha \in C$.

We observe that the distribution $G_\alpha(P \pm io,m,n)$ is a causal (anticausal) analogue of the kernel due to N.Aronszajn - K.T. Smith and A.P.Calderón (cf. [1] and [2] , respectively).
The particular radial case of our problem was solved by Nogin, for $\alpha \neq 1,2,3,\ldots$ (cf. [3]).

I. INTRODUCTION

Let $x = (x_1,x_2,\ldots,x_n)$ be a point of the n-dimensional Euclidean space R^n. Consider a non-degenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 , \qquad (I,1)$$

where $n = p+q$. The distribution $(P \pm io)^\lambda$ is defined by

$$(P \pm io)^\lambda = \lim_{\varepsilon\to 0} \{P \pm i\varepsilon|x|^2\}^\lambda , \qquad (I,2)$$

where $\varepsilon > 0$, $|x|^2 = x_1^2 + \ldots + x_n^2$, $\lambda \in C$.

The distributions $(P \pm io)^\lambda$ are analytic in λ everywhere except at $\lambda = -\frac{n}{2} - k$, $k = 0,1,\ldots$; where they have simple poles (cf. [4] , p.275).

The distributions $(m^2 + Q \pm io)^\lambda$ are defined in an analogue manner as the distributions $(P \pm io)^\lambda$. Let us put (cf. [4], p.289)

$$(m^2 + Q \pm io)^\lambda = \lim_{\varepsilon \to 0} (m^2 + Q \pm i\varepsilon |y|^2)^\lambda \quad , \qquad (I,3)$$

where m is a positive real number, $\lambda \in \mathbf{C}$, ε is an arbitrary positive number. In the formula (I,3) we have written

$$Q = Q(y) = y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_{p+q}^2 \quad , \qquad (I,4)$$

$p + q = n$,

and $$|y|^2 = y_1^2 + \dots + y_n^2 \; .$$

It is useful to state an equivalent definition of the distributions $(m^2 + Q \pm io)^\lambda$.

In this definition appear the distributions

$$(m^2 + Q)_+^\lambda \begin{array}{ll} = (m^2 + Q)^\lambda & \text{if } m^2 + Q \geqslant 0 \; , \\ = 0 & \text{if } m^2 + Q < 0 \; . \end{array} \qquad (I,5)$$

$$(m^2 + Q)_-^\lambda \begin{array}{ll} = 0 & \text{if } m^2 + Q > 0 \; , \\ = (-m^2 - Q)^\lambda & \text{if } m^2 + Q \leqslant 0 \; . \end{array} \qquad (I,6)$$

We can prove, without difficulty, that the following formula is valid (cf. [7], p.566)

$$(m^2 + Q \pm io)^\lambda = (m^2 + Q)_+^\lambda + e^{\pm i\pi\lambda}(m^2 + Q)_-^\lambda \; . \quad (I,7)$$

From this formula we conclude immediately that

$$(m^2 + Q + io)^\lambda = (m^2 + Q - io)^\lambda = (m^2 + Q)^\lambda \; , \quad (I,8)$$

when $\lambda = k$ = positive integer.

We observe that $(m^2 + Q \pm io)^\lambda$ are entire distributional functions of λ.

Let $G_\alpha(P \pm io, m, n)$ be the causal (anticausal) distribution defined by

$$G_\alpha(P \pm io,m,n) = H_\alpha(m,n)(P \pm io)^{\frac{1}{2}(\frac{\alpha-n}{2})} K_{\frac{n-\alpha}{2}}[m(P \pm io)] \quad , \quad (I,9)$$

where m is a positive real number, $\alpha \in C$, K_μ designates the well-known modified Bessel function of the third kind (cf. [5], p.78, formulae (6) and (7)):

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\text{sen } \pi\nu} \quad , \qquad (I,10)$$

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m+\nu}}{m! \; \Gamma(m+\nu+1)} \qquad (I,11)$$

and $H_\alpha(m,n)$ is the constant defined by

$$H_\alpha(m,n) = \frac{e^{\pm\frac{\pi}{2}qi} \; e^{i\frac{\pi}{2}\alpha} \; 2^{1-\frac{\alpha}{2}} \; (m^2)^{\frac{1}{2}(\frac{n-\alpha}{2})}}{(2\pi)^{\frac{n}{2}} \; \Gamma(\frac{\alpha}{2})} \; . \qquad (I,12)$$

The following formula is valid (cf. [6], p.35, formula (II,1.8)):

$$[G_\alpha(P \pm io,m,n)]^\Lambda = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{i\pi\frac{\alpha}{2}} (m^2 + Q \pm io)^{-\frac{\alpha}{2}} \; . \qquad (I,13)$$

Here Λ denotes the Fourier transform of a distribution.

We observe that the right-hand member of (I,13) is an entire distribution of α; therefore G_α is also an entire distributional function of α.

II. THE PROPERTIES OF $G_\alpha(P \pm io,m,n)$

The Bessel potential of order α (α being any complex number) of a temperate distribution f, denoted by $J^\alpha f$ is defined by

$$(J^\alpha f)^\Lambda = (1 + 4\pi^2|x|^2)^{-\frac{\alpha}{2}} f^\Lambda \quad , \qquad (II,1)$$

was introduced by N.Aronszajn - K.T.Smith and A.P.Calderón (cf. [1] and [2], respectively).

A.P.Calderón proves in [2], Theorem 1, that

$$J^{\alpha} f = G_{\alpha} * f \quad , \qquad \text{(II,2)}$$

where

$$G_{\alpha} = G_{\alpha}(x) = \gamma(\alpha)\ e^{-|x|} \int_0^{\infty} e^{-|x|t} \left(t + \frac{t^2}{2}\right)^{\frac{n-\alpha-1}{2}} dt \quad , \qquad \text{(II,3)}$$

Re $\alpha < n+1$, and

$$[\gamma(\alpha)]^{-1} = (2\pi)^{\frac{n-1}{2}}\ \Gamma\left(\frac{\alpha}{2}\right)\ \Gamma\left(\frac{n-\alpha+1}{2}\right) . \qquad \text{(II,4)}$$

We start by observing that the distributional function $G_{\alpha}(P \pm io,m,n)$ (cf. formula (I,9)) is an (causal, anticausal) analogue of the kernel defined by the formula (II,3).

The distributions $G_{\alpha} = G_{\alpha}(P \pm io,m,n)$ share many properties with the Bessel kernel of which they are (causal, anticausal) analogues.

The following theorems hold:

THEOREM II.1. *Let us put* $\alpha \in C$, $k = 0,1,\dots$, *then*

$$\{G_{\alpha} * G_{-2k}\}^{\Lambda} = (2\pi)^{\frac{n}{2}}\ \{G_{\alpha}\}^{\Lambda} . \{G_{-2k}\}^{\Lambda} . \qquad \text{(II,5)}$$

Here * designates, as usual, the convolution.

THEOREM II.2. *The following formula is valid*

$$G_{\alpha} * G_{-2k} = G_{\alpha-2k} \quad , \qquad \text{(II,6)}$$

when $\alpha \in C$, $k = 0,1,2,\dots$.

More generally, the following formulae are valid for all $\alpha,\beta \in C$,

$$G_o(P \pm io,m,n) = \delta \quad , \qquad \text{(II,7)}$$

$$\{G_\alpha * G_\beta\}^\Lambda = (2\pi)^{\frac{n}{2}} \{G_\alpha\}^\Lambda . \{G_\beta\}^\Lambda \quad , \qquad (II,8)$$

and

$$G_\alpha * G_\beta = G_{\alpha+\beta} \; . \qquad (II,9)$$

Let us define the n-dimensional ultrahyperbolic Klein-Gordon operator, iterated ℓ-times:

$$K^\ell = \left\{ \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} - m^2 \right\}^\ell =$$

$$= \{\Box - m^2\}^\ell \quad , \qquad (II,10)$$

where $p+q = n$, $m \in R^+$, $\ell = 1,2,\ldots$.

From the preceding results we deduce the explicit expression of a family of elementary (causal, anticausal) solution of the ultrahyperbolic Klein-Gordon operator, iterated k-times.

In fact, the following proposition is valid.

THEOREM II.3. *The distributional functions* $G_{2k}(P \pm io,m,n)$ *where* n = *integer* $\geqslant 2$ *and* $k = 1,2,\ldots$, *are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator, iterated* k-*times*:

$$K^k \{G_{2k}(P \pm io,m,n)\} = \delta \; . \qquad (II,11)$$

The proofs of the formulae (II,5), (II,6), (II,7), (II,8), (II,9) and (II,11) appear in [6].

It may be observed that the elementary solutions $G_{2k}(P \pm io,m,n)$ have the same form for all $n \geqslant 2$. This does not happen for other elementary solution, whose form depends essentially on the parity of n (cf. [7], p.580 and [8], p.403).

We observe that the particular case of Theorem II.3 corresponding to n=4, k=1, q=1 is especially important.

The corresponding elementary solutions can be written

$$G_2(P+io,m,4) = -\frac{mi}{4\pi^2}\,\frac{K_1[m(P+io)^{1/2}]}{(P+io)^{1/2}}\,, \qquad \text{(II,12)}$$

$$G_2(P-io,m,4) = \frac{mi}{4\pi^2}\,\frac{K_1[m(P-io)^{1/2}]}{(P-io)^{1/2}}\,. \qquad \text{(II,13)}$$

The formula (II,12) is a useful expression of the famous "magic function" or "causal propagator" of Feynman.

For this reason we have decided to call "causal" ("anticausal") the distributions $G_\alpha(P \pm io,m,n)$.

III. THE INVERSE ULTRAHYPERBOLIC BESSEL KERNEL

Let $B^\alpha f$ be the ultrahyperbolic Bessel operator defined by the formula

$$B^\alpha f = G_\alpha * f\,, \qquad \text{(III,1)}$$

$f \in S$.

Our objective is the attainment of $T^\alpha = (B^\alpha)^{-1}$ such that if $\varphi = B^\alpha f$, then $T^\alpha \varphi = f$.

We note that the inverse ultrahyperbolic Bessel kernel $(B^\alpha)^{-1}$ is, formally, by virtue of (I,13) and (II,10), a fractional power of the differential Klein-Gordon operator:

$$(B^\alpha)^{-1} = (\Box - m^2)^{\frac{\alpha}{2}}\,. \qquad \text{(III,2)}$$

Therefore, here we are seeking an explicit expression for $(B^\alpha)^{-1}$. The following theorem expresses that if we put, by definition,

$$B^\alpha = G_\alpha\,, \qquad \text{(III,3)}$$

then

$$(B^\alpha)^{-1} = (G_\alpha)^{-1} = G_{-\alpha} \qquad \text{(III,4)}$$

for all complex α.

Now we shall state our main theorem.

THEOREM III.1. *If*

$$\varphi = B^{\alpha}f \quad , \tag{III,5}$$

where $B^{\alpha}f$ *is defined by* (III,1), $f \in S$; *then*

$$T^{\alpha}\varphi = f \quad , \tag{III,6}$$

where

$$T^{\alpha} = (B^{\alpha})^{-1} = G_{-\alpha} \tag{III,7}$$

$\alpha \in C$.

Here G_{α} *is defined by* (I,9) *and* α *being any complex number.*

Proof. From the definitory formula (III,1) we have

$$B^{\alpha}f = G_{\alpha} * f = \varphi \quad , \tag{III,8}$$

where G_{α} is given by (I,9), $\alpha \in C$ and $f \in S$.
Then, in view of (II,9) and (II,7), we obtain

$$G_{-\alpha} * (G_{\alpha} * f) = (G_{-\alpha} * G_{\alpha}) * f = G_{-\alpha+\alpha} * f =$$
$$= G_{0} * f = \delta * f = f \; . \tag{III,9}$$

Therefore

$$G_{-\alpha} = (B^{\alpha})^{-1} \tag{III,10}$$

Formula (III,10) is the desired result and this finished the proof of Theorem III.1 ∎

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