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AN APPROXIMATION THEOREM FOR CERTAIN SUBSETS OF SOBOLEV SPACES

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SUMMARY. We show that a class of differentiable functions vanishing together with their derivatives of order less than r on the boundary of a smooth domain Ω is dense in the subset of $W^{m+r,p}(\Omega)$ defined by the functions already in $W_o^{r,p}(\Omega)$. We give a direct proof by introducing a particular extension operator and a related reflection operator. These subsets are Banach spaces that we call $W^p_{r,m+r}(\Omega)$.

1. PRELIMINARIES AND NOTATION. Let Ω be a domain in R^n. By $(.,.)$ and $\|.\|$ we shall always denote the scalar product and norm in $L^2(\Omega)$. For r a nonnegative integer we denote by $H^r(\Omega)$ the Sobolev space $H^r(\Omega) := \{u \in D'(\Omega);\ D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leqslant r\}$ with the norm $\|u;H^r(\Omega)\| = (\sum_{|\alpha|\leq r} \|D^\alpha u\|^2)^{1/2}$ and by $\mathring{H}^r(\Omega)$ the closure of $C_o^\infty(\Omega)$ in $H^r(\Omega)$ (cfr. [A] where $H^r(\Omega) = W^{r,2}(\Omega)$ and $\mathring{H}^r(\Omega) = W_o^{r,2}(\Omega)$). We state some well known facts about these spaces that we shall need in what follows.

LEMMA 1. *If* $u \in H^r(\Omega)$, $v \in \mathring{H}^r(\Omega)$ *and* $|\alpha| \leqslant r$, *then*

$$(D^\alpha u, v) = (u, D^\alpha v).$$

Proof. If $v_h \in C_o^\infty(\Omega)$ is a sequence such that $\|v_h - v;H^r(\Omega)\| \to 0$

then

$$(D^\alpha u,v) = \lim_{h\to\infty} (D^\alpha u,v_h) = \lim_{h\to\infty} (u,D^\alpha v_h) = (u,D^\alpha v), \quad \text{Q.E.D.}$$

Let Ω be a bounded domain with C^∞ boundary (i.e. there exists a finite open covering of $\partial\Omega$, $\{U_j; j = 1,\dots,N\}$, such that for each j there is a map ϕ_j from U_j onto $B = \{y \in R^n; |y| < 1\}$ with the properties: i) ϕ_j is one to one, ii) $\phi_j \in C^\infty(U_j)$, $\phi_j^{-1} \in C^\infty(B)$, iii) $\phi_j(U_j \cap \Omega) = B^+ = \{y \in B; y_n > 0\} = B \cap R_n^+$).

LEMMA 2. *If* $u \in C^r(\bar{\Omega})$ *and* $D^\alpha u = 0$ *on* $\partial\Omega$ *for* $|\alpha| < r$, *then* $u \in \overset{\circ}{H}{}^r(\Omega)$.

Proof. Let U_o be an open subset of Ω such that $\bigcup_{j=0}^{N} U_j \supset \bar{\Omega}$. Using a C^∞ partition of unity subordinate to this covering one sees that it is enough to prove that: if $u \in C^r(\overline{R_n^+})$, $D^\alpha u(x_1,\dots,x_{n-1},0) = 0$ for $|\alpha| < r$ and supp u is bounded, then $u \in \overset{\circ}{H}{}^r(R_n^+)$, (cf. [A], T.3.35, particularly formula (15)). Now in that case let $\tilde{u}(x) := u(x)$ for $x \in R_n^+$ and 0 otherwise. Then Gauss' theorem yields for $\phi \in C_o^\infty(R_n)$ and $|\alpha| \leqslant r$

$$\int_{R_n} (-1)^{|\alpha|}\tilde{u}D^\alpha\phi\, dx = (-1)^{|\alpha|}\int_{R_n^+} uD^\alpha\phi\, dx = \int_{R_n^+} D^\alpha u.\phi dx = \int_{R_n} \widetilde{D^\alpha u}.\phi dx$$

That is, $D^\alpha\tilde{u}$ is the function $\widetilde{D^\alpha u}$ for $|\alpha| \leqslant r$ and so $\tilde{u} \in H^r(R_n)$. But then $u = \lim_{\varepsilon\to 0} v_\varepsilon$ in $H^r(R_n^+)$ where $v_\varepsilon(x) = \tilde{u}(x_1,\dots,x_{n-1},x_n-\varepsilon)$. Since supp v_ε is compact in R_n^+, $v_\varepsilon \in \overset{\circ}{H}{}^r(R_n^+)$ and the proof is complete, Q.E.D.

2. INTRODUCTION. For r, R positive integers, $r < R$, let us call $H_{r,R}(\Omega)$ the Hilbert space $H_{r,R}(\Omega) := \overset{\circ}{H}{}^r(\Omega) \cap H^R(\Omega)$ with the norm of $H^R(\Omega)$ and call $D_r(\Omega) := \{\phi \in C^\infty(\bar{\Omega}); D^\alpha\phi = 0 \text{ on } \partial\Omega \text{ for } |\alpha| < r\}$.

Now let Ω be a bounded domain with C^{∞} boundary. By Lemma 2, $D_r(\Omega) \subset H_{r,R}(\Omega)$. (It also follows that this space contains properly the space $\overset{\circ}{H}{}^R(\Omega)$, cf.Th.5). In this paper we prove that $D_r(\Omega)$ is a dense subset of $H_{r,R}(\Omega)$. That is

THEOREM 1. *If* $G_{r,R}(\Omega)$:= *closure of* $D_r(\Omega)$ *in* $H^R(\Omega)$, *then*

$$G_{r,R}(\Omega) = H_{r,R}(\Omega).$$

This theorem can be proved in the particular case $R = 2r$ using results of P.D.E. as follows. For $\lambda > 0$ the operator $(-\Delta)^r + \lambda$ maps $H_{r,2r}(\Omega)$ continuously into $L^2(\Omega)$. This map is also 1:1 since for $u \in H_{r,2r}(\Omega)$ using Lemma 1 we obtain

$$((-\Delta)^r u + \lambda u, u) = \sum_{|\alpha|=r} (r!/\alpha!)(D^{2\alpha}u, u) + \lambda \|u\|^2 =$$

$$= \sum_{|\alpha|=r} (r!/\alpha!)\|D^{\alpha}u\|^2 + \lambda \|u\|^2.$$

On the other hand for λ great enough $((-\Delta)^r + \lambda)G_{r,2r} = L^2(\Omega)$ (cfr. [S], Th.9-27, pg.219). In consequence $G_{r,2r}(\Omega) = H_{r,2r}(\Omega)$. We shall give a direct proof of this fact and moreover of Theorem 1. By using a partition of unity as in Lemma 2 it is enough to prove

THEOREM 2. *Let* K *be a compact set in* B *and* $u \in H_{r,R}(R^+_n)$ *with* supp $u \subset K \cap R^+_n$. *Then there exists a sequence* $u_h \in D_r(R^+_n)$ *such that* supp $u_h \subset B^+$ *and* $\|u_h - u; H^R(R^+_n)\| \to 0$ *for* $h \to \infty$.

Our proof relies on the following result.

3. AUXILIARY LEMMA. *Given* R *integers* $K_1, K_2, \ldots, K_R$ *there exists a polynomial* $p(x)$ *of degree* $R-1$ *such that*

i) $p(2^j)$ *is an integer for* $j = 0,1,\ldots$

ii) $p(2^{m-1}) = K_m$ *(mod 2) for* $1 \leqslant m \leqslant R$

iii) $p(2^{m-1}) = K_R$ (*mod* 2) *for* $R < m$.

Proof. If $x_i = 2^{i-1}$, $i = 1,2,\ldots,R$, define $p(x)$ by

$$p(x) := \sum_{j=1}^{R} h_j \prod_{k=1}^{j-1} ((x - x_k)/x_k) \cdot \prod_{k=j+1}^{R} ((x - x_k)/x_j)$$

where $h_j = 0$ if K_j is even and $h_j = 1$ if K_j is odd. Observe that $p(x)$ satisfies i) and ii) since $(x_j - x_k)/x_s$ is odd for $s = \min(j,k)$ and is even for $s < \min(j,k)$. By the same reasoning for $x = x_m$, $m > R$, the first product in the definition of p is odd and the last is even when not empty. So $p(x_m) - p(x_R)$ is even, Q.E.D.

COROLLARY. *Given* R *integers* $K_1,\ldots,K_R$, *there exists an entire function* $f(z)$ *without zeroes such that*

i) $f(2^{j-1}) = (-1)^{K_j}$ *for* $j = 1,\ldots,R$

ii) $f(2^{j-1}) = 1/f(2^{j-1})$ *for* $j \in N$.

Proof. Define

(1) $$f(z) := \exp(i\pi p(z))$$

where $p(z)$ is the polynomial in the preceding lemma. Then both $f(z)$ and $g(z) := 1/f(z)$ have the required properties, Q.E.D.

4. AN EXTENSION OPERATOR. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be an entire function. We associate to f the operator

(2) $$(T_f u)(x',t) := \sum_{k=0}^{\infty} c_k u(x',-2^k t)$$

where $x' = (x_1,\ldots,x_{n-1}) \in R_{n-1}$, $t \in R_1$.

T_f is well defined if u vanishes outside a sphere.

THEOREM 3. *For* $u \in H^R(R_n^+)$, supp $u \subset B^+$, *we have*

i) supp $T_f u \subset B^-$, $T_f u \in H^R(R_n^-)$

ii) $\| T_f u; H^R(R_n^-) \| \leq M(f) \; \| u; H^R(R_n^+) \|$ *and*

(3) $$D^\alpha T_f u = T_{f_h}(D^\alpha u) \text{ for } h = \alpha_n, \; |\alpha| \leq R$$

where f_h *is the entire function*

(4) $$f_h(z) := (-1)^h \sum_{k=0}^{\infty} c_k 2^{h.k} z^k = (-1)^h f(2^h z).$$

Proof. The first assertion of i) is immediate. The second follows from ii). Observe that if $x = (x',t) \in K$, a compact set in R_n^-, then the sum defining $T_f u(x',t)$ is finite. Therefore (3) is correct in $D'(R_n^-)$. To prove ii) it is therefore enough to prove

(5) $$\| T_{f_h} u; L^2(R_n^-) \| \leq M(f_h) \| u; L^2(R_n^+) \|.$$

But $\| u(x',-2^k t) \| = 2^{-k/2} \; \| u \|$. Summing up, one gets

$$\| T_{f_h} u \| \leq \left(\sum_{k=0}^{\infty} |c_k| . 2^{k(2h-1)/2} \right) \| u \| \qquad \text{Q.E.D.}$$

Observe that the lemma remains true if the roles of R_n^+ and R_n^- are interchanged. Now we define the extension operator E_f associated to $f(z) = \sum c_k z^k$ by

(6) $$E_f u(x',t) := \begin{cases} u(x',t) & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ T_f u(x',t) & \text{for } t < 0. \end{cases}$$

THEOREM 4. *Let* $u \in H_{r,R}(R_n^+)$, *closure in* R_n *of* supp $u \subset B$. *If the entire function* $f(z)$ *verifies*

(7) $$f(2^s) = (-1)^s \quad \textit{for} \quad s = r, r+1, \ldots, R-1$$

then $E_f u \in H^R(R_n)$, supp $E_f u \subset B$ *and*

(8) $$\| E_f u; H^R(R_n) \| \leq C(f) \; \| u; H^R(R_n^+) \|.$$

Proof. We shall show that if $|\alpha| \leqslant R$, $h = \alpha_n$ and f_h is defined by (4), then

$$D^{\alpha}(E_f u) = E_{f_h}(D^{\alpha}u). \tag{9}$$

Therefore, the theorem will follow from Th.3. To prove (9) we consider two cases.

CASE 1: $\alpha = (0,\dots,0,h)$. Let $\phi \in C_o^{\infty}(R_n)$. Then if we set $x = (x',t)$,

$$\langle D^{\alpha}E_f u,\phi\rangle = (-1)^h\langle E_f u, D_t^h\phi\rangle = \tag{10}$$

$$= (-1)^h\int_{R_n^+}\Big(uD_t^h\phi + (-1)^h\sum_{k=0}^{\infty} c_k u(x',2^k t).D_t^h\phi(x',-t)\Big)\,dx =$$

$$= (-1)^h\int_{R_n^+} uD_t^h\phi dx + \sum_{k=0}^{\infty} 2^{(h-1)k}c_k\int_{R_n^+} u(x)D_t^h(\phi(x',-2^{-k}t))dx =$$

$$= (-1)^h\int_{R_n^+} u(x)D_t^h\psi_h(x',t)dx'dt$$

with

$$\psi_h(x',t) = \phi(x',t) - \sum_{k=0}^{\infty}(-2^k)^{h-1}c_k\phi(x',-2^{-k}t). \tag{11}$$

Since $\sum_{k=0}^{\infty}|c_k|.\,M^k < \infty$ for any $M > 0$, it is possible to interchange $\sum$ and $\int$ in (10). Also $\psi_h \in C_o^{\infty}(\overline{R_n^+}) \cap H^s(R_n^+)$ for any s.

Now we shall show that

$$(-1)^h\int_{R_n^+} u(x)D_t^h\psi_h(x',t)dx = \int_{R_n^+} D_t^h u.\psi_h dx. \tag{12}$$

In fact, since $u \in \mathring{H}^r(R_n^+)$, by Lemma 1,

$$(-1)^h\int_{R_n^+} uD_t^h\psi_h dx = (-1)^{h-j}\int_{R_n^+} D_t^j u.D_t^{h-j}\psi_h dx \quad \text{for } j = \min(h,r). \tag{13}$$

This proves (12) for $h \leq r$. If $h > r$, then in view of (7), $\psi_h(x',0) = 0$ and also $D^\gamma \psi_h(x',0) = 0$ for $|\gamma| < h-r$. Then by Lemma 2, $\psi_h \in \overset{\circ}{H}{}^{h-r}(R_n^+)$, and we can apply again Lemma 1 to the right hand side of (13) ($j = r$ now!) thus obtaining (12). The combination of (10) with (12) yields

$$<D^\alpha E_f u,\phi> = \int D^\alpha u.\psi_h \, dx = <E_{f_h}(D^\alpha u),\phi>.$$

CASE 2: $\alpha = (\alpha_1,\ldots,\alpha_{n-1},0)$. Then (9) is true regardless condition (7) for $u \in H^q(R_n^+)$, $q \geq |\alpha|$. In fact, let $\eta(t) \in C^\infty(R_1)$, $\eta = 0$ for $|t| < 1/2$, $\eta = 1$ for $|t| > 1$, and call $\eta_\varepsilon(t) := \eta(t/\varepsilon)$. Then for $\phi \in C_o^\infty(R_n)$ we have $\eta_\varepsilon\phi \in C_o^\infty(R_n^- \cup R_n^+)$ and so

$$<D^\alpha E_f u,\phi> = (-1)^{|\alpha|}<E_f u,D^\alpha\phi> = (-1)^{|\alpha|}\lim_{\varepsilon\to 0}<E_f u,\eta_\varepsilon D^\alpha\phi> =$$

$$= \lim_{\varepsilon\to 0}(-1)^{|\alpha|}<E_f u,D^\alpha(\eta_\varepsilon\phi)> = \lim_{\varepsilon\to 0}<E_f(D^\alpha u),\eta_\varepsilon\phi> = <E_f(D^\alpha u),\phi>.$$

To combine this two cases we write $\alpha = (\alpha_1,\ldots,\alpha_{n-1},0) + (0,\ldots,0,h) = \alpha' + \alpha''$ and obtain $D^\alpha(E_f u) = D^{\alpha'}E_{f_h}(D^{\alpha''}u) = E_{f_h}(D^\alpha u)$, Q.E.D.

5. A REFLECTION OPERATOR. Next we define an operator E which is a generalization of $\phi(x',t) \to -\phi(x',-t)$. Let $f(z) = \sum_{k=0}^\infty c_k z^k$ be the entire function constructed in the Corollary of section 3 for $K_i = i$ if $i \leq r$ and $K_i = i-1$ for $r < i \leq R$.

That is

(14) $f(2^{i-1}) = (-1)^i$ for $1 \leq i \leq r$; $f(2^{i-1}) = (-1)^{i-1}$ for $r < i \leq R$.

Further, let $g(z) := 1/f(z) = \sum_{k=0}^\infty d_k z^k$. For v a function with bounded support let us define

$$Ev(x',t) := \begin{cases} T_g v = \sum_{k=0}^{\infty} d_k v(x',-2^k t) & \text{for} \quad t > 0, \\ T_f v = \sum_{k=0}^{\infty} c_k v(x',-2^k t), & \quad t \leq 0. \end{cases}$$

LEMMA 3. *If* $\phi \in C_o^\infty(R_n)$, *then*

i) $E\phi \in C_o^\infty(R_n)$

ii) $E\phi = \phi$ *implies* $\phi \in D_r(R_n^+)$

iii) $E^2\phi = \phi$

iv) $\| E\phi ; H^s(R_n) \| \leq M_s \| \phi ; H^s(R_n) \| \quad \forall\ s \in N.$

v) *Let* $v \in H^s(R^n)$, *support of* $v \subset B$. *If the sequence* $\{\phi_m\} \subset C_o^\infty(B)$ *verifies* $\lim_{m\to\infty} \| \phi_m - v ; H^s(R^n) \| = 0$, *then*

$$\lim_{m\to\infty} \| E\phi_m - Ev ; H^s(R^n) \| = 0.$$

Proof. i) It is clear from the definition that supp $E\phi$ is bounded and that $E\phi \in C^\infty(R_n^- \cup R_n^+)$. Also

$$(15) \quad \begin{cases} D^\alpha E\phi(x',+0) = (\sum_{k=0}^{\infty} d_k (-2^k)^{\alpha_n}) D^\alpha \phi(x',0) = (-1)^{\alpha_n} g(2^{\alpha_n}) D^\alpha \phi(x',0) \\ D^\alpha E\phi(x',-0) = (\sum_{k=0}^{\infty} c_k (-2^k)^{\alpha_n}) D^\alpha \phi(x',0) = (-1)^{\alpha_n} f(2^{\alpha_n}) D^\alpha \phi(x',0). \end{cases}$$

i) then follows from

$$(16) \qquad f(2^h) = g(2^h) = \pm 1.$$

ii) Let $\alpha_n < r$. Using (15) and (14) it follows that

$$(17) \qquad D^\alpha E\phi(x',0) = (-1)^{\alpha_n} f(2^{\alpha_n}) D^\alpha \phi(x',0) = -D^\alpha \phi(x',0).$$

But if $E\phi = \phi$ then

$$(18) \qquad D^\alpha E\phi(x',0) = D^\alpha \phi(x',0)$$

Comparing (17) and (18) we get $D^\alpha \phi(x',0) = 0$ for $|\alpha| < r$, that is $\phi \in D_r(R_n^+)$.

iii) Observe that $T_g T_f \phi(x',t) = \sum_{k=0}^{\infty} d_k (\sum_{h=0}^{\infty} c_h \phi(x', 2^{k+h}t)) =$

$= \sum_{j=0}^{\infty} \phi(x', 2^j t) (\sum_{k=0}^{j} d_k c_{j-k})$.

Since $f(z).g(z) = 1$ we have $\sum_{k=0}^{j} d_k c_{j-k} = 1$ if $j=0$ and 0 otherwise. Therefore, it holds pointwise that

$$(19) \qquad T_g T_f \phi(x',t) = \phi(x',t) = T_f T_g \phi(x',t).$$

iv) By i), $\|E\phi;H^s(R_n)\| \leqslant \|T_f\phi;H^s(R_n^-)\| + \|T_g\phi;H^s(R_n^+)\|$. Now Theorem 3 yields iv).

v) By iv), $E\phi_m$ is a Cauchy sequence in $H^s(R^n)$. Therefore, there exists $U \in H^s(R^n)$ such that $\|E\phi_m - U;H^s(R^n)\|$ tends to zero.
But in virtue of Theorem 3, ii) both norms $\|E\phi_m - T_g v;H^s(R_n^+)\|$ and $\|E\phi_m - T_f v;H^s(R_n^-)\|$ tend to zero. So U restricted to R_n^+ is equal to $T_g v$ and U restricted to R_n^- is $T_f v$. Since the distribution U is a function of $L^2(R^n)$ it follows that $U = Ev$, Q.E.D. Note that conditions (14) for $r < i \leqslant R$ are not really used in the proof of Lemma 3.

6. **PROOF OF THEOREM 2.** Let $u \in H_{r,R}(R_n^+)$, supp $u \subset K$ and call $u' := E_f u$ (cfr.(6)). Observe that by (14) the hypotheses of Theorem 4 are fulfilled. Thereby $u' \in H^R(R_n)$, supp $u' = K' =$ = compact in B and $Eu' \in H^R(R_n)$. In consequence, from the definition of u' we have $Eu' = u'$ a.e. (cf.(19)). Now let $\phi_h' \in C_o^\infty(B)$ be a sequence converging to u' in $H^R(R_n)$. By Lemma 3, v), $E\phi_h'$ converges to $Eu' = u'$ in $H^R(R_n)$ and then

(20) $$\|u' - \phi_h;H^R(R_n)\| \to 0 \text{ for } h \to \infty$$

if $\phi_h := (\phi'_h + E\phi'_h)/2$.

Using Lemma 3, iii), we see that $E\phi_h = \phi_h$. Then by ii) of the same Lemma we obtain that $u_h := \phi_h$ restricted to R_n^+ belongs to $D_r(R_n^+)$. Since $\|u - u_h;H^R(R_n^+)\| = \|u' - \phi_h;H^R(R_n^+)\| \leq \leq \|u' - \phi_h;H^R(R_n)\|$, we see by (20) that the sequence u_h satisfies all the requirements, Q.E.D.

7. THE SPACES $W^p_{r,R}(\Omega)$. Our method can be applied to prove that $D_r(\Omega)$ is dense in other Banach spaces. For $1 \leq p < \infty$, $0 < r < R$, r, R integers, let us define

$$W^p_{r,R}(\Omega) := W^{r,p}_o(\Omega) \cap W^{R,p}(\Omega) \text{ with the norm } \|.;W^{R,p}\|.$$

THEOREM 1'. *If* Ω *is a bounded domain with* C^∞ *boundary, then* $D_r(\Omega)$ *is dense in* $W^p_{r,R}(\Omega)$.

This theorem reduces to prove

THEOREM 2'. $\{u \in D_r(R_n^+)$: supp u bounded$\}$ *is dense in* $W^p_{r,R}(R_n^+)$.

The proof follows the same lines as that of Theorem 2 noticing that the operator E_f defined by (6) is continuous from $W^p_{r,R}(R_n^+)$ into $W^{R,p}(R_n)$, and the operator E of Lemma 3 is continuous in $W^{R,p}(R_n)$. Lemma 2 should be replaced by

LEMMA 2'. *If* $u \in C^r(\bar{\Omega})$ *and* $D^\alpha u = 0$ *on* $\partial\Omega$ *for* $|\alpha| < r$, *then* $u \in W^{r,p}_o(\Omega)$.

THEOREM 5. *Let* r *be a positive integer and* R *a nonnegative one. The completion of* $D_r(\Omega)$ *in the norm* $\|.;W^{R,p}(\Omega)\|$ *is isomorphic*

to the space $W_o^{R,p}(\Omega)$ *if* $R \leq r$ *and isomorphic to* $W^p_{r,R}(\Omega) \supsetneq$
$\supsetneq W_o^{R,p}(\Omega)$ *if* $R > r$.

Proof. In fact, for $R \leq r$, because of Lemma 2', we have

$$C_o^\infty(\Omega) \subset D_r'(\Omega) \subset D_R(\Omega) \subset W_o^{R,p}(\Omega).$$

If $R > r$, it follows from Theorem 1' that $W^p_{r,R} \supset W_o^{R,p}$. To prove that the inclusion is proper consider the function $k(x) = x_n^r\phi(x')\psi(x_n)$ restricted to R_n^+ where $\phi(x') \in C_o^\infty(R_{n-1})$, $\psi \in C_o^\infty(R_1)$, ϕ and ψ equal to one in a neighborhood of zero. Then, k is of bounded support and belongs to $W^{R,p}(R_n^+) \cap W_o^{r,p}(R_n^+)$. If k belonged to $W_o^{R,p}(R_n^+)$ then $\tilde{k}$ should belong to $W^{R,p}(R_n)$. However, $D_{x_n}^{r+1}\tilde{k}$ is not a function, Q.E.D.

By the same argument one gets, for $r < S < R$, the proper inclusions

$$W_o^{r,p} \supsetneq W^p_{r,S} \supsetneq W^p_{r,R} \supsetneq W_o^{R,p}. \tag{21}$$

It also holds, since Ω is bounded, that the norm

$$\Big(\sum_{j=r}^{R} \sum_{|\alpha|=j} \|D^\alpha u; L^p(\Omega)\|^p\Big)^{1/p} \tag{22}$$

is equivalent to the original norm in $W^p_{r,R}(\Omega)$, (cf.[A], p.158).

8. COMMENTS. The construction of the extension operator (6), E_f, with f as in paragraph 3, is similar to the one used by Seeley in [Se] however corresponding to entire functions of different nature. In order that E_f extends $C^\infty(\overline{R_n^+})$ to $C^\infty(R_n)$, Seeley needs $f(2^h) = (-1)^h$ for $h = 0,1,\ldots$ and this is not true for our f since we have $f(2^h) = (-1)^{h+1}$ for $h = 0,\ldots,r-1$ (on the other hand the coefficients a_k found by Seeley define

an entire function of exponential type with zeroes and in that case $g = 1/f$ is not entire). This explains the main difference between our extension operator and that of Seeley and other extension operators, for example, the altogether different one constructed by A.P.Calderón ([C], p.45). It consists in the fact that for the extension E_f the functions $D^{\alpha}E_f u$, $|\alpha| < r$, can be discontinuous at the boundary except in the case when they vanish there, and therefore E_f does not define a continuous operator from $W^{R,p}(\Omega)$ into $W^{R,p}(R_n)$ (but it does when restricted to $W^p_{r,R}(\Omega)$, (cf.Th.4)).

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