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BOUNDEDNESS OF SINGULAR INTEGRALS ON NON NECESSARILY NORMALIZED SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. We consider the question whether it is possible to study singular integrals on spaces of homogeneous type without normalizing the metric. An affirmative answer is given, provided the measure satisfies a smoothness condition which is strictly less restrictive than de ones considered by Aimar [A] and David, Journé and Semmes [DJS]. To attain this we develop useful results on the structure of spaces of homogeneous type.

INTRODUCTION

The study of problems in the setting of spaces of homogeneous type has been proven to be very fruitful in order to obtain general results which have a wide range of applications. Frequently it seems unavoidable to know a quantitative relation between the measure of the ball and its radius especially in order to get boundedness results involving integrals. This is usually attained assuming the space to be normal as for instance, in [A], [CW], [DJS] and [W]. Essentially a space is normal if the measure of a ball is comparable to its radius. If the open balls are open sets then the space can be normalized as it is proven in [MS2]. However the resulting metric is

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not equivalent to the original one. Also in many instances some kind of smoothness condition on the measure of the balls needs to be considered as in [A] and [DJS], see [Ca] for an example in a different situation.

We pose ourselves the question up to which point these restrictions are necessary. That is, for instance, whether it is possible to give conditions for the boundedness of singular integrals on L^2, following Cotlar's lemma approach, preserving the original metric. Which allows to keep truck of the shape of the truncations. In a normal space conditions were given in [A], [DJS] and [W]. We show in Theorem A that an affirmative answer can be given considering truncations adapted to the measure of the balls rather than to their radius, provided we impose a smoothness condition on the measure (see (1.7)), which is strictly less restrictive than the ones adopted in [A] and [DJS]. Through several examples we discuss its meaning in §4. We should point out that in [W] no smoothness condition is assumed, however this work contains some serious errors, see our remark (2.23). In order to achieve our purpose we are led to make in §2 a careful study of integration problems, which we believe to be of independent interest and usefulness when working on spaces of homogeneous type.

§1. MAIN RESULTS

Before stating the results we recall some basic facts and definitions.

A quasi-distance on a set X is a non negative symmetric function, d(x,y) defined on X x X such that d(x,y) = 0 if and only if x=y, and there exists a constant K satisfying

$$(1.1) \qquad d(x,y) \leq K[d(x,z)+d(z,y)] ,$$

for every x,y and z in X.

As it is proved in [MS2] given a quasi-distance d there exist a quasi-distance d', uniformly equivalent to d, a finite constant C and a number β, $0 < \beta \leq 1$, such that

$$|d'(x,y)-d'(x,z)| \leqslant C\, r^{1-\beta}\, d'(y,z)^{\beta} ,$$

provided $d'(x,y) \leqslant r$ and $d'(x,z) \leqslant r$. Therefore we assume, without loss of generality, the existence of $0 < \beta \leqslant 1$ such that

$$(1.2) \qquad |d(x,y)-d(x,z)| \leqslant K\, d(x,y)^{1-\beta}\, d(y,z)^{\beta} ,$$

holds whenever $d(x,y) \geqslant d(x,z)$.

The sets $\{(x,y) \in X \times X: d(x,y) < 1/n\}$, $n > 0$, define a basis of a metrizable uniform structure on X. The balls

$$B(x,r) = \{y \in X: d(x,y) \leqslant r\}$$

form a basis of neighborhoods for the topology induced by the uniform structure. We call the attention on the fact that $B(x,r)$ is not being used to denote the set

$$B^{o}(x,r) = \{y \in X: d(x,y) < r\} ,$$

as it is usual. It can be seen that $B(x,r)$ is not in general the closure of $B^{o}(x,r)$. As will become apparently the sets $B(x,r)$ are more adjusted to our purposes.

A space of homogeneous type is a set X endowed with a quasi-distance $d(x,y)$ and a Borel measure μ satisfying a "doubling condition" i.e. there exists a constant A such that

$$(1.3) \qquad \mu(B(x,2Kr)) \leqslant A\mu(B(x,r)) ,$$

for every $r > 0$. We say that the space of homogeneous type is normal if there exist finite and positive constants A_1, A_2, K_1 and K_2 such that

$$\begin{aligned} A_1 r \leqslant \mu(B(x,r)) \leqslant A_2 r \quad &, \quad \text{when } K_2\mu(\{x\}) \leqslant r \leqslant K_1\mu(X) \\ B(x,r) = X \quad &, \quad \text{if } r > K_1\mu(X) \text{ and} \\ B(x,r) = \{x\} \quad &, \quad \text{if } r < K_2\mu(\{x\}) . \end{aligned}$$

To assume that X is normal provides a knowledge of the measure of a ball given its radius. This fact was basic, when doing integration in papers dealing with L^2 boundedness of singular integrals like [A], [DJS] and [W]. We want to avoid any hypothesis of this type. Hence we use a "smoothness" condition on the measure alternative to the ones adopted in [A] and [DJS].

The space of homogeneous type X is said to satisfy property H if there exists α, $0 < \alpha \leq 1$, such that

$$(1.4)\quad \mu(B(x,r+r^{1-\beta}s^{\beta}))-\mu(B(x,r-r^{1-\beta}s^{\beta})) \leq A\mu(B(x,r))^{1-\alpha}\mu(B(x,s))^{\alpha},$$

holds for every $x \in X$ and $0 < s \leq r$, where β is the constant appearing in (1.2).

Since, for instance, assuming H does not eliminate spaces containing points with positive measure (1.4) is strictly less restrictive than those conditions considered in [A] and [DJS]. We present in §4 some examples which will help to clarify the meaning of hypothesis H.

The standard hypothesis on the singular integral kernel $K(x,y)$ must also be modified in order to avoid hypothesis of the normalization type. We shall study singular integral kernels $K(x,y)$ satisfying the following

(K.1) $|K(x,y)| \leq C\mu(B(x,d(x,y)))^{-1}$, for every $x \neq y$.

(K.2) There exists α, $0 < \alpha \leq 1$ such that for every integer $k \geq 1$

$$|K(x,y)-K(x,z)| + |K(y,x)-K(z,x)| \leq CA^{-\alpha k}\mu(B(y,d(x,y)))^{-1},$$

provided $A^k\mu(B(y,d(y,z))) \leq \mu(B(y,d(y,x)))$.

(K.3) Let $0 < r < R < \infty$

$$\int_{r<\mu(B(x,d(x,y)))\leq R} K(x,y)\,d\mu(y) = 0, \text{ for every } x \in X$$

$$\int_{r<\mu(B(x,d(x,y)))\leq R} K(x,y)\,d\mu(x) = 0, \text{ for every } y \in X.$$

Given $0 < r < R < \infty$, define

$$K_{r,R}f(x) = \int_{r<\mu(B(x,d(x,y)))<R} K(x,y)f(y)\,d\mu(y).$$

The L^2 result can be stated as follows.

THEOREM A. *Let* X *be a space of homogeneous type with property* H. *Assume the kernel* K(x,y) *satisfies* (K.1), (K.2) *and* (K.3) *then there exists* C *such that*

$$\|K_{r,R}f\|_2 \leqslant C\|f\|_2$$

for every $0 < r < R$, *and* $f \in L^2(X)$.

Having into account lemma (2.2) below it can be seen that condition (K.2) is weaker than

$$\text{(K.2)'} \quad |K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \leqslant \frac{Cd(z,y)}{d(x,y)\mu(B(y,d(x,y)))}$$

provided $2d(z,y) < d(y,x)$.

In order to simplify the notation and since no confusion arises in the following we shall write $|E|$ instead of $\mu(E)$ and dx instead of $d\mu(x)$.

§2. TECHNICAL RESULTS

LEMMA (2.1). *Assume* $0 < d(z,y) < [K(2K)^{1-\beta}]^{-\beta}r$. *Then*

$$\emptyset \neq B(z,r-\delta r^{1-\beta}d(y,z)^{\beta}) \subset B(y,r) \subset B(z,r+\delta r^{1-\beta}d(z,y)^{\beta}),$$

for every δ *such that* $K(2K)^{1-\beta} \leqslant \delta < r^{\beta}d(y,z)^{-\beta}$.

Proof. By the assumptions $r-\delta r^{1-\beta}d(y,z)^{\beta} > 0$. Take x in $B(z,r-\delta r^{1-\beta}d(y,z)^{\beta})$, then $d(x,y) < 2Kr$. Applying (1.2)

$$d(x,y) \leqslant d(x,z)+K(2Kr)^{1-\beta}d(y,z)^{\beta} \leqslant r.$$

Then $x \in B(y,r)$. By another application of (1.2) we obtain the remaining inclussion.

LEMMA (2.2). *If* $A^j\,|B(y,s)| < |B(y,r)|$, *then* $(2K)^j s < r$.

Proof. It follows immediately from the doubling property (1.3).

LEMMA (2.3). *Assume* X *satisfies property* H, (1.4), *then either* $|\{x\}| > 0$ *for every* $x \in X$ *or* $|\{x\}| = 0$ *for every* $x \in X$.

Proof. If there exists $z \in X$ of measure zero, then given any $x \in X$ and $\delta > 0$

$$\begin{aligned} |\{x\}| &\leq |B(z,d(z,x)+\delta^{\beta}d(z,x)^{1-\beta})| - \\ &- |B(z,d(z,x))-\delta^{\beta}d(z,x)^{1-\beta}| \leq \\ &\leq A|B(z,d(z,x))|^{1-\alpha} B(z,\delta)|^{\alpha}. \end{aligned}$$

Letting δ tend to zero, we obtain $|\{x\}| = 0$.

DEFINITION (2.4). Let $r > 0$; we denote $E(x,r) = \{y:|B(x,d(x,y))| \leq r\}$.

LEMMA (2.5). *Let* $R = \sup \{d(x,y): y \in E(x,r)\}$, *then*

(2.6) $B^{o}(x,R) \subset E(x,r) \subset B(x,R)$.

(2.7) $|E(x,r)| \leq r$.

(2.8) $|B(x,R)| \leq Ar$.

(2.9) *If* $|B(x,R)| \leq r$ *then* $E(x,r) = B(x,R)$.

(2.10) *If* $|B(x,R)| > r$ *then* $E(x,r) = B^{o}(x,R)$.

Proof. As it is well known if $|X| < \infty$ then X is bounded, on the other hand if $|X| = \infty$ then $|B(x,s)|$ goes to infinity if s tends to infinity, therefore $R < \infty$. If y belongs to $E(x,r)$ by definition of R, $d(x,y) \leq R$, in particular

$$E(x,r) \subset B(x,R).$$

Let us assume $|B(x,R)| \leq r$. If $y \in B(x,R)$ then

$$|B(x,d(x,y))| \leq |B(x,R)| \leq r.$$

This says that $y \in E(x,r)$, in particular $E(x,r) = B(x,R)$. Thus (2.9) is proved and (2.6), (2.7) and (2.8) in this case.

Assume now $|B(x,R)| > r$. Therefore $d(x,y) < R$ for every y in $E(x,r)$ and there exists a sequence $\{y_j\}_j$ such that $y_j \in E(x,r)$ and $r_j = d(x,y_j)$ increases to R. Then

$$E(x,r) = \bigcup_{j=1}^{\infty} B(x,r_j) = B^{o}(x,R)$$

and

$$|E(x,r)| = \lim_{j\to\infty} |B(x,r_j)| \leq r.$$

This proves (2.10) and completes the proof of (2.6), (2.7) and (2.8).

LEMMA (2.11). $|B(y,d(z,y))| < A|B(z,d(z,y))| < A^2|B(y,d(z,y))|$.

Proof. It is immediate from the doubling property (1.3) and (1.1).

With the notation of Lemma (2.5), we have

LEMMA (2.12). *If* X *satisfies* H *then*

$$E(x,r) = B(x,R) ,$$

for every x *in* X *and* $r > 0$.

Proof. Let us assume $|\{x\}| = 0$. Then given $0 < \eta < R$,

$$0 \leq |B(x,R)| - |B^o(x,R)| \leq |B(x,R+R^{1-\beta}\eta^\beta)| - |B(x,R-R^{1-\beta}\eta^\beta)| \leq$$
$$\leq A|B(x,R)|^{1-\alpha}|B(x,\eta)|^\alpha.$$

Therefore $|B(x,R)| \leq |B^o(x,R)| \leq r$. Thus $E(x,r) = B(x,R)$. Let us suppose $|\{x\}| > 0$. Then by lemma (2.3), $|\{y\}| > 0$ for every $y \in X$. Thus $\{y\} = B(y,\varepsilon)$ for some $\varepsilon = \varepsilon(y) > 0$ (see [MS2]). By (2.9) we can assume $|B(x,R)| > r$. In this case there exists a sequence $\{y_j\}_j$ such that $y_j \in E(x,r)$ and $r_j = d(x,y_j)$ increases to R. Let $0 < \varepsilon_j < r_j$ satisfying $B(y_j,\varepsilon_j) = \{y_j\}$. Thus

$$0 < |\{x\}| \leq |B(y_j,r_j+r_j^{1-\beta}\varepsilon_j^\beta)| - |B(y_j,r_j-r_j^{1-\beta}\varepsilon_j^\beta)| \leq$$
$$\leq A|B(y_j,r_j)|^{1-\alpha}|B(y_j,\varepsilon_j)|^\alpha.$$

By (2.11) this is less than

$$A^2|B(x,r_j)|^{1-\alpha}|\{y_j\}|^\alpha \leq A^2\, r^{1-\alpha}|\{y_j\}|^\alpha.$$

This implies the existence of an infinite number of points contained in B(x,R) with measure greater than a constant which is impossible.

COROLLARY (2.13). *If* X *satisfies* H *and* $|\{x\}| = 0$ *for some* $x \in X$, *then*

$$|B(y,r)| = |B^{o}(y,r)|$$

for any $y \in X$ *and* $r > 0$.

Let us denote $C(x,k) = E(x,A^{k+1}) \setminus E(x,A^{k})$.

LEMMA (2.14). *Assume* $|X| \geqslant A^{k+1} \geqslant |\{x\}|$. *Then* $C(x,k) \neq \emptyset$ *for every* $x \in X$.

Proof. Let $N_j = \{r\colon A^j < |B(x,r)| \leqslant A^{j+1}\}$. Assume $N_j \neq \emptyset$ and let $r \in N_j$. Since

$$B = B(x,r) = \bigcup_{y \in B} B(x,d(x,y))$$

there exists y such that

$$A^j < |B(x,d(x,y))| \leqslant A^{j+1} ,$$

then $C(x,j) \neq \emptyset$. Therefore it is enough to prove $N_k \neq \emptyset$. In order to do this we shall show that $N_k = \emptyset$ implies $N_j = \emptyset$ for every $j > k$. Let $k_1 = \min\{j > k\colon N_j \neq \emptyset\}$ and r_1 be the infimum of N_{k_1}. Thus, $r_1 > 0$ and

$$A^{k_1} \leqslant |B(x,r_1)| \leqslant A^{k_1+1} .$$

If $|B(x,r_1)| = A^{k_1}$ then $r_1 \in N_{k_1-1}$ which is a contradiction. Therefore r_1 must belong to N_{k_1}. On the other hand, $(2K)^{-1}r_1 \notin N_{k_1}$ implying

$$|B(x,r_1)| \leqslant A|B(x,(2K)^{-1}r_1)| \leqslant AA^{k} \leqslant A^{k_1},$$

which is again a contradiction.

In the following we assume X satisfies property H. Let

$$R_k^x = \sup\{d(x,y)\colon y \in E(x,A^k)\}.$$

By lemma (2.12), $E(x,A^k) = B(x,R_k^x)$. In particular,

$$C(x,k) = B(x,R_{k+1}^x)-B(x,R_k^x). \tag{2.15}$$

Moreover by (2.7) and Lemma (2.14), if $|X| \geq A^{k+1} \geq |\{x\}|$,

$$A^{k-1} < |B(x,R_k^x)| \leq A^k. \tag{2.16}$$

LEMMA (2.17). *Let us suppose* $(2K)^h \geq [2K(2K)^{1-\beta}]^{1/\beta}$. *Then*

$$A^h|B(x,d(x,z))| < A^{i-1},$$

implies

$$C(x,i)\Delta C(z,i) \subset [B(x,R_{i+1}^x+S_{i+1}^x)-B(z,R_{i+1}^x-S_{i+1}^x)] \cup$$
$$\cup [B(z,R_i^x+S_i^x)-B(z,R_i^x-S_i^x)] \ ,$$

where $$S_j^x = \frac{3}{2} K(2K)^{1-\beta}(R_j^x)^{1-\beta}d(x,z)^\beta \ .$$

Proof. By (2.16) and the assumption

$$A^h|B(x,d(x,z))| < A^{i-1} < |B(x,R_i^x)|.$$

Thus using Lemma (2.2)

$$[2K(2K)^{1-\beta}]^{1/\beta}d(x,z) \leq (2K)^h d(x,z) < R_i^x \leq R_{i+1}^x. \tag{2.18}$$

Then we can apply Lemma (2.1) with $\delta = \frac{3}{2} K(2K)^{1-\beta}$ to obtain

$$B(z,R_j^x-S_j^x) \subset B(x,R_j^x) \subset B(z,R_j^x+S_j^x), \quad j = i,i+1 \ ,$$

where $$S_j^x = \delta(R_j^x)^{1-\beta} \ d(x,z)^\beta.$$

Having into account (2.15) if $y \in C(x,i)\Delta C(z,i)$ we have four possibilities

P_1: $y \in C(x,i)-B(z,R_{i+1}^z)$. Then $y \in B(z,R_{i+1}^x+S_{i+1}^x)-B(z,R_{i+1}^z)$.
P_2: $y \in C(x,i) \cap B(z,R_i^z)$. Then $y \in B(z,R_i^z)-B(z,R_i^x-S_i^x)$.
P_3: $y \in C(z,i)-B(z,R_{i+1}^x)$. Then $y \in B(z,R_{i+1}^z)-B(z,R_{i+1}^x-S_{i+1}^x)$.
P_4: $y \in C(z,i) \cap B(x,R_i^x)$. Then $y \in B(z,R_i^x+S_i^x)-B(z,R_i^z)$.

The proof is complete.

COROLLARY (2.19). *On the conditions of the Lemma above*

$$|B(z,R_i^x - S_i^x)| > A^{i-3}.$$

Moreover if $y \in C(x,i)\Delta C(z,i)$ *then*

$$|B(z,d(y,z))| > A^{i-4}.$$

Proof. By (2.18) $d(x,z) < R_i^x$ and $4(R_i^x - S_i^x) > R_i^x$. Thus using (2.16) and the doubling condition of the measure

$$A^{i-1} < |B(x,R_i^x)| < A|B(z,R_i^x)| < A^3|B(z,R_i^x - S_i)|.$$

On the other hand if $y \in C(z,i)$ then $|B(z,d(z,y))| > A^i$. If $y \in C(x,i)-C(z,i)$ then either P_1 or P_2 hold. In the first case $y \notin E(z,A^{i+1})$ therefore $|B(z,d(z,y))| > A^{i+1}$. In the second case $d(z,y) > R_i^x - S_i^x$ therefore

$$|B(z,d(z,y))| \geq |B(z,R_i^x - S_i^x)| > A^{i-4}.$$

COROLLARY (2.20). *In the conditions of Lemma* (2.17)

$$|C(x,i)\Delta C(z,i)| \leq CA^{i(1-\alpha)}|B(z,d(x,z))|^{\alpha}.$$

Proof. Property H implies

$$D_i = |B(z,R_i^x + \eta^{\beta} R_i^{x(1-\beta)}) - B(z,R_i^x - \eta^{\beta} R_i^{x(1-\beta)})| \leq$$
$$\leq C|B(z,R_i^x)|^{1-\alpha}|B(z,\eta)|^{\alpha}.$$

From (2.18) it follows that we can apply Lemma (2.1).

$$B(z,R_i^x) \subset B(x,R_i^x + \delta R_i^{x(1-\beta)}\, d(x,z)^{\beta}) \subset$$
$$\subset B(x,2R_i^x).$$

Therefore taking $\eta^{\beta} = \frac{3}{2} K(2K)^{1-\beta} d(x,z)^{\beta}$ and appling doubling property

$$D_i \leqslant C|B(x,R_i^x)|^{1-\alpha}|B(z,d(x,z))|^{\alpha} \leqslant$$
$$\leqslant CA^{i(1-\alpha)}|B(z,d(x,z))|^{\alpha}.$$

Clearly the same bound is true for D_{i+1}.

Let $\chi_i(x,y) = \chi_{C(x,i)}(y)$. Observe that

(2.21) $\chi_i(x,y) \leqslant \chi_{i-1}(y,x)+\chi_i(y,x)+\chi_{i+1}(y,x)$.

LEMMA (2.22). *If* $A^i < |X|$, *it follows*

$$I(i,q,x) = \int |B(x,d(x,y))|^q \chi_i(x,y)dy \leqslant CA^{i(1+q)}$$
$$I'(i,q,y) = \int |B(x,d(x,y))|^q \chi_i(x,y)dx \leqslant CA^{|q|}A^{i(1+q)}.$$

If either $A^i \geqslant |X|$ *or* $|\{x\}| \geqslant A^i$ *both integrals are zero.*

Proof.
$$\int |B(x,d(x,y))|^q \chi_i(x,y)dy \leqslant CA^{iq}|E(x,A^{i+1})| \leqslant CA^{i(1+q)}$$

Let us consider $I'(i,q,y)$, by Lemma (2.11) and (2.21)

$$I'(i,q,y) \leqslant A^{|q|} \int |B(y,d(x,y))|^q \chi_i(x,y)dx \leqslant$$
$$\leqslant A^{|q|} [I(i-1,q,y)+I(i,q,y)+I(i+1,q,y)] \leqslant$$
$$\leqslant CA^{|q|} A^{i(1+q)}.$$

REMARK (2.23). Lemma (2.22) seems to be a substitute for inequality (1) in Theorem (2.1) of [W]. That inequality is not true in general as can be seen taking $X = \{0,1,2,3,\dots,n,\dots\}$; $\mu(n) = 1$; $d(n,m) = |n-m|$ and considering f to be $\chi_{[0,1]} \leqslant f < \chi_{[0,2]}$ or $\chi_{[1,\infty)} \leqslant f < \chi_{[0,\infty)}$.

§ 3. PROOFS OF THEOREMS

We shall use the following

COTLAR'S LEMMA. *Let* H *be a Hilbert space and* $T_1, T_2, \dots, T_N$ *a finite of linear and continuous operators on* H. *Let* $c: \mathbf{Z} \to [0,\infty)$ *such that* $\sum_{\ell=-\infty}^{\infty} c(\ell)^{1/2} = A < \infty$ *and let* T_i^* *be the adjoint of* T_i. *If* $\|T_i^* T_j\| \leqslant c(i-j)$ *and* $\|T_i T_j^*\| \leqslant c(i-j)$ *then* $\|\sum_{i=1}^{N} T_i\| \leqslant A$.
See [C].

Proof of Theorem A. We denote $K_j(x,y) = K(x,y)\chi_j(x,y)$ and define $T_j f(x) = \int K_j(x,y) f(y)\,dy$, for any function f in $L^2(X)$. Using Lemma (2.22) and proceeding as in [A] it follows that T_j is uniformly bounded on L^2 and the kernel of the adjoint operator T_j^* is $K_j^*(x,y) = K_j(y,x)$. Moreover

$$T_i^* T_j f(x) = \int \{ \int K_i(y,x) K_j(y,z)\,dy \} f(z)\,dz.$$

and

$$|T_i^* T_j f(x)|^2 \leqslant \int \left| \int K_i(y,x) K_j(y,z)\,dy \right| |f(z)|^2\,dz \quad \times$$
$$\times \int \left| \int K_i(y,x) K_j(y,z)\,dy \right|\,dz .$$

Applying again Lemma (2.22) and the boundedness of the kernel the second factor is bounded by a constant independent of i and j. Assume that, whenever $j > i$

$$\int \left| \int K_i(y,x) K_j(y,z)\,dy \right|\,dz \leqslant CA^{\alpha(i-j)}. \tag{3.1}$$

Then, in this case

$$\|T_i^* T_j f\|_2^2 \leqslant CA^{\alpha(i-j)} \int |f(z)|^2 [\int \left| \int K_i(y,x) K_j(y,z)\,dy \right| dx]\,dz$$
$$\leqslant CA^{\alpha(i-j)} \|f\|_2^2 .$$

If $i > j$, using again (3.1)

$$\|T_i^*T_jf\|_2^2 \leq C\int |f(z)|^2 [\int|\int K_i(y,x)K_j(y,z)dy|dx]\,dz \leq$$
$$\leq CA^{\alpha(j-i)}\|f\|_2^2 .$$

It is clear that the same estimate holds for $\|T_iT_j^*\|_2^2$. Therefore aplying Cotlar's Lemma the Theorem follows.

Let us prove (3.1). By (K.3)

$$I = \int|\int K_i(y,x)K_j(y,z)dy|dz =$$
$$= \int|\int [K_j(y,z)-K_j(x,z)]\, K_i(y,x)dy|dz \leq$$
$$\leq \int|K_i(y,x)|\{\int|K_j(y,z)-K_j(x,z)|dz\}dy .$$

Take $C_1 = A^h$, where $(2K)^h \geq [2K(2K)^{1-\beta}]^{1/\beta}$. Let

$$I_1 = \int_{C_1|B(y,d(y,x))|\geq A^{j-1}} |K_i(y,x)|\{\int|K_j(y,z)-K_j(x,z)|dz\}dy .$$

By (K.1) and Lemma (2.22)

$$\int|K_j(y,z)|dz \leq C\int_{C(y,j)} \frac{1}{|B(y,d(y,z)|}\,dz \leq C.$$

Also

$$\int|K_j(x,z)|dz \leq C.$$

Therefore using once more (K.1) and Lemma (2.22)

$$I_1 \leq C\int_{C_1|B(y,d(y,x))|\geq A^{j-1}} \frac{\chi_i(y,x)}{|B(y,d(y,x)|}\,dy \leq$$
$$\leq C\int [\frac{A^j}{|B(y,d(y,x))|}]^{-\alpha}\,\frac{\chi_i(y,x)}{|B(y,d(y,x))|}\,dy \leq$$
$$\leq CA^{\alpha(i-j)} .$$

It remains to consider

$$I_2 = \int_{C_1|B(y,d(y,x))|<A^{j-1}} |K_i(y,x)|\{\int |K_j(y,z)-K_j(x,z)|dz\}dy.$$

In order to prove the boundedness of I_2 let

$$I_{21} = \int_{C_1|B(y,d(y,x))|<A^{j-1}} |K_i(y,x)|\{\int |K(y,z)-K(x,z)|\chi_j(y,z)dz\}dy.$$

Observe that if $x \in C(y,i)$, $z \in C(y,j)$ and $A^h|B(y,d(x,y))| < A^{j-1}$,

$$A^{\ell}|B(y,d(x,y))| \leq 1 < A^{-j}|B(y,d(y,z))|,$$

where $\ell = \max(h-j+1,-i-1)$. Then

$$A^{j+\ell}|B(y,d(x,y))| < |B(y,d(y,z))|.$$

Therefore using (K.2) and (K.1)

$$I_{21} \leq C \int \frac{\chi_i(y,x)}{|B(y,d(y,x))|} \{\int A^{-\alpha(j-i)} \frac{\chi_j(y,z)}{|B(y,d(y,z))|} dz\} dy.$$

Applying Lemma (2.22)

$$I_{21} \leq CA^{-\alpha(j-i)}.$$

We need to show a similar bound for I_{22}

$$I_{22} = \int_{C_1|B(y,d(y,x))|<A^{j-1}} |K_i(y,x)|\{\int |\chi_j(y,z)-\chi_j(x,z)||K(x,z)|dz\}dy$$

$$\leq C \int_{C_1|B(y,d(y,x))|<A^{j-1}} \frac{\chi_i(y,x)}{|B(y,d(x,y))|}\{\int_{C(y,j)\Delta C(x,j)} \frac{1}{|B(x,d(x,z))|} dz\} dy.$$

By Corollary (2.19) we get

$$I_{22} \leq CA^{-j} \int_{C_1|B(y,d(y,x))|<A^{j-1}} \frac{\chi_i(y,x)}{|B(y,d(x,y))|} |C(y,j)\Delta C(x,j)|dy.$$

By Corollary (2.20) and Lemma (2.22)

$$I_{22} \leqslant CA^{-j}A^{j(1-\alpha)} \int \chi_i(y,x)\,|B(y,d(x,y))|^{\alpha-1}\,dy \leqslant$$

$$\leqslant CA^{-\alpha(j-i)}.$$

Therefore

$$I_2 \leqslant I_{21} + I_{22} \leqslant CA^{-\alpha(j-i)}.$$

§4. EXAMPLES

We present some examples in order to understand better property H and the relation between the metric and the measure. In previous works, besides the requirement for X to be normalized, the following hypothesis has being imposed

$$(4.1) \qquad |B(x,r)| - |B(x,s)| \leqslant C(r-s)^{\gamma} r^{1-\gamma},$$

for every $0 \leqslant s < r$ and some $\gamma > 0$. The case $\gamma=1$ has being considered in [A] and γ positive in [DJS]. Observe that the condition is really meaningful when s is close to r, since otherwise the normalization of the space suffices. Thus, it is clear that (4.1) is more restrictive as γ increases, therefore it is enough to consider $\gamma \leqslant 1$. Clearly (4.1) implies H. The contrary is not true even in the case of normalized X; see example (4.5). Thus, all the examples considered in [A] are included, as for instance $\mathbf{R}^n$ with parabolic metrics and Lebesgue measure, compact groups with a quasi-metric induced by a Vitali family, etc.

EXAMPLES OF SPACES THAT DO NOT SATISFY H.

(4.2) Let $X = [R - (2,4)] \cup \{3\}$, $d(x,y)$ the usual distance and measure μ defined by $\mu(\{3\}) = 2$ and $\mu(E - \{3\}) = |E|$, where $|E|$ stands for Lebesgue measure. It is easy to see that (X,d,μ) is a space of homogeneous type that does not satisfy the conclusion of Lemma (2.3).

(4.3) For $n > 0$, let I_n be the open interval in $\mathbf{R}$ centered at

$2n+\frac{1}{2}$ of radius $(2n)^{-1}$. Let

$$X = \bigcup_{n=1}^{\infty}\{[2n-1,2n] \cup I_n\}$$

Take $d(x,y)$ the euclidean distance and define the measure μ

$$\mu(E) = \sum_{n=1}^{\infty} \int \chi_E(x)(\chi_{[2n-1,2n]}(x)+n\chi_{I_n}(x))\, dx.$$

It is clear that (X,d,μ) is an space of homogeneous type. Moreover

$$1 \leqslant \mu(B(2n,\tfrac{1}{2}+\tfrac{1}{2n})) - \mu(B(2n,\tfrac{1}{2}-\tfrac{1}{2n}))$$

holds for every $n > 0$. On the other hand

$$\mu(B(2n,\tfrac{1}{2}))^{1-\alpha}\ \mu(B(2n,\tfrac{1}{2n}))^{\alpha} \to 0$$

as n tends to infinity for any $0 < \alpha \leqslant 1$.

EXAMPLES OF SPACES SATISFYING H.

(4.4) The space considered in (2.23) satisfies H.

(4.5) Let (X,d^{γ},μ) be a space of homogeneous type, where d is distance, $\gamma > 0$ and such that

$$\mu(B(x,r)) = r^{\lambda}\ ,\ \lambda > 0.$$

It is well known, see for instance [MS 1], that if

$$\beta = \min(1,\tfrac{1}{\gamma})$$

then d^{γ} satisfies (1.2). We shall prove that property H is true for $\alpha = \min(1,\beta/\lambda)$. We observe that it is sufficient to take $r > 2\eta > 0$ on property H. Let $\varepsilon = r^{1-\beta}\eta^{\beta}$

$$D = \mu(B(x,r+\varepsilon)) - \mu(B(x,r-\varepsilon)) = (r+\varepsilon)^{\lambda}-(r-\varepsilon)^{\lambda} =$$

$$= \lambda\ \xi^{\lambda-1}\ 2\varepsilon,$$

where $r-\varepsilon < \xi < r+\varepsilon$. From $r > 2\eta > 0$ it follows

$$D \leqslant C\lambda r^{\lambda-1}2\varepsilon = C(r^{\lambda})^{1-\beta/\lambda}\eta^{\beta} \leqslant C(r^{\lambda})^{1-\alpha}(\eta^{\lambda})^{\alpha}.$$

(4.6) Let $(X_i, d_i^{\gamma_i}, \mu_i)$, i=1,2, as in (4.5). It is clear that taking $X = X_1 \times X_2$, $\mu = \mu_1 \otimes \mu_2$ and

$$d(\bar{x},\bar{y}) = d((x_1,x_2),(y_1,y_2)) = \max\{d_1^{\gamma_1}(x_1,y_1), d_2^{\gamma_2}(x_2,y_2)\},$$

(X,d,μ) becomes a space of homogeneous type. Moreover it can be shown that

$$\mu(B(\bar{x},r)) = r^{\lambda_1+\lambda_2},$$

$$|d(\bar{x},\bar{y}) - d(\bar{x},\bar{z})| \leqslant Cr^{1-\beta}d(\bar{y},\bar{z})^{\beta},$$

for $d(\bar{x},\bar{y}) < r$; $d(\bar{x},\bar{z}) < r$ and $\beta = \min(\beta_1,\beta_2)$. Proceeding as in (4.5) it follows that (X,d,μ) satisfies H, with

$$\alpha = \min(1,\tfrac{\beta}{\lambda}) = \min\left(1,\frac{\min(\beta_1,\beta_2)}{\lambda_1+\lambda_2}\right).$$

(4.7) As an immediate application of (4.6) we consider

$$R^n = R^{n_1} \times \ldots \times R^{n_k}, \quad n_i \geqslant 1,$$

$$d_i(x_i,y_i) = |x_i - y_i|^{\gamma_i}, \quad \gamma_i > 0,$$

μ_i = Lebesgue measure in R^{n_i}, thus $\mu_i(B(x_i,r)) = C_i\, r^{n_i/\gamma_i}$. Therefore taking $\lambda_i = n_i/\gamma_i$ and $\beta_i = \min(1,\gamma_i^{-1})$, by (4.5), we get

$$\alpha_i = \min(1,\frac{\beta_i}{\lambda_i}) = \frac{\min(1,\gamma_i)}{n_i}.$$

Then we are in the conditions of (4.6) and (X,d,μ) satisfies H with

$$\lambda = \sum_{i=1}^{k} \frac{n_i}{\gamma_i}; \quad \beta = \min(1,\frac{1}{\gamma_1},\ldots,\frac{1}{\gamma_k})$$

$$\alpha = \min(1,\frac{\beta}{\lambda}) = \frac{\min(1,\gamma_1^{-1},\ldots,\gamma_k^{-1})}{\gamma_1^{-1}n_1+\ldots+\gamma_k^{-1}n_k} .$$

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