Revista de la
Unión Matemática Argentina
Volumen 34, 1988.

ROUND QUADRATIC FORMS

ENZO R. GENTILE

A regular quadratic form ψ over a field K of characteristic $\neq 2$ is called *round* if either it is hyperbolic or it is anisotropic, satisfying the following similarity conditions: for any $x \in K$ represented by ψ, the isometry $<x>\psi \approx \psi$, holds.

We shall study in this Note, round forms over a linked field mainly with u-invariant $u(K) \leqslant 4$. If $\dim \psi = 2^v.\ell$, $v \geqslant 2$, ℓ odd, this study was made by M.Marshall [M]. We here complete it to forms of dimensions 2ℓ, ℓ odd. We rely on results in [M].

1. PRELIMINARIES.

K will denote a field of characteristic $\neq 2$. Quadratic forms over K will be regular (i.e.non-degenerate) and written in diagonal form $<a_1,\dots,a_n>$, $a_i \in K$. If ψ and φ are quadratic forms, $\psi \perp \varphi$ denotes orthogonal sum and $\psi.\varphi$, tensor product. If ψ is a quadratic form, with $D(\psi)$ we denote the set of all elements of K represented by ψ and $\dot{D}(\psi) := D(\psi)\setminus\{0\}$.

For $a_1,\dots,a_n \in K$, the n-fold Pfister form $<1,a_1>.<1,a_2> \dots <1,a_n>$ will be denoted by $<<a_1,a_2,\dots,a_n>>$. A non-empty subset T of K will be called a *preordering* if it is closed respect to sums and products, i.e. if $T+T \subset T$ and $T.T \subset T$.

A (quadratic) form ψ over K will be called *round* if either, i) ψ is hyperbolic or ii) ψ is anisotropic and for all $x \in D(\psi)$, $x \neq 0$, $\langle x\rangle.\psi \simeq \psi$, i.e. the similarity factors of ψ coincide with $\dot{D}(\psi)$.

A field K is called *linked* (or, a *linked field*) if the classes of quaternion algebras over K form a subgroup in the Brauer group of K. We shall use results on linked fields contained in [G_1] or [G_2].

The u-invariant of a field K is by definition: $u(K) = \max\{\dim q\}$ where q runs over the anisotropic torsion forms over K. If K is a linked field then it is well-known that $u(K) \in \{0,1,2,4,8\}$.

With W(K) we shall denote the witt ring of K, consisting of the witt classes of all quadratic forms.

Next, we recall some basic results that will be needed in this paper. Both are from [M]. We give a proof of Proposition 1 which avoids the use of the Hasse-invariant.

1.1. PROPOSITION ([M], Prop.1.1 (ii)). *Let ψ be a round form of dimension $2.\ell$, ℓ odd. Then*

$$D(\psi) \subset D(\langle 1,\det\psi\rangle).$$

Proof. Let $a \in D(\psi)$, so $\langle a\rangle.\psi \simeq \psi$, and hence $\langle 1,-a\rangle.\varphi = 0$ in the ring of K. This means that $\varphi \in \mathrm{Ann}(\langle 1,-a\rangle)$ =: annihilator ideal in W(K) of $\langle 1,-a\rangle$. Now it is well known (see [EL], Cor. 2.3) that we have an isometry

$$\psi \simeq \beta_1 \perp \ldots \perp \beta_\ell ,$$

where $\beta_i = \langle c_1\rangle.\langle 1,-b_1\rangle$, $b_i \in D(\langle 1,-a\rangle)$. Therefore $-\det\psi =$ $= b_1 \ldots b_\ell \in D(\langle 1,-a\rangle)$.

Thus $\langle 1,-a\rangle = \langle -\det\psi,\det\psi.a\rangle$

or $\langle 1,\det\psi\rangle = \langle a,\det\psi.a\rangle$,

and so $a \in D(\langle 1,\det\psi\rangle)$.

1.2 PROPOSITION ([M], Prop.2.7). *Let* K *be a linked field with* $u(K) \leqslant 4$. *Let* ψ *be a round form over* K *of dimension* $2^{\nu}\ell$, ℓ *odd. Then*

i) *If* $\nu = 2$, *there exists a unique* ν*-fold Pfister form* ψ_0 *defined over* K *such that* $\psi \simeq \ell \times (\langle\det\psi\rangle \oplus \psi_0')$ $(\psi_0 \simeq \langle 1\rangle \perp \psi_0')$.

ii) *if* $\nu \geqslant 3$, *there exists a unique* ν*-fold Pfister form* ψ_0 *and a unique universal 2-fold Pfister form* ρ *defined over* K *such that* $\psi \simeq \ell \times (\langle\det\psi\rangle \oplus \psi_1')$ *where* ψ_1 *is defined by* $\psi_0 \oplus \rho \simeq \psi \oplus 2H$. (H *denotes a hyperbolic plane*).

2. ROUND FORMS OVER LINKED FIELDS.

2.1. PROPOSITION. *Let* K *be a linked field and* ψ *a round form of dimension* 2ℓ, $\ell > 1$, *odd. Then*

i) $\psi \simeq \perp \langle x_i\rangle\varphi_i \perp \langle 1,\det\psi\rangle$ *with* φ_i, *2-fold Pfister forms,* $x_i \in K$;

ii) $q := \perp \langle x_i\rangle\varphi_i$ *is a round form and* $D(\psi) = D(q)$;

iii) $D(\psi) = D(\langle 1,\det\psi\rangle) = D(q)$;

iv) $D(\psi)$ *is a preordering.*

Proof. If $\ell = 1$ then $\psi = \langle 1,a\rangle$, and so we can assume $\ell > 1$.

i) Being K a linked field we can write (see $[G_1]$)

$$\psi \simeq \perp \langle y_i\rangle.\varphi_i \perp \langle a,b\rangle$$

with φ_i, 2-fold Pfister forms.

Clearly, $\det\psi = a.b$. If we multiply ψ by $\langle a\rangle$ we get

$$\psi \simeq \langle a\rangle\psi \simeq \perp \langle x_i\rangle\varphi_i \perp \langle 1,\det\psi\rangle.$$

ii) From Prop.1.1 we have $D(\psi) \subset D(\langle 1,\det\psi\rangle)$ and then from i) it is clear that

$$D(\psi) = D(\langle 1,\det\psi\rangle).$$

iii) is consequence of i) and ii).

iv) Let $x,y \in D(\psi)$. Then $x \in D(\perp \langle x_i\rangle\varphi_i)$ and $y \in D(\langle 1,\det\psi\rangle)$

and so $x+y \in D(\psi)$. For the product $x.y$, it is clear that $x.y \in D(\psi)$.

2.2. PROPOSITION. *Let* K *be a linked field with* $u(K) \leqslant 4$ *and let* ψ *be a round form of dimension* 2ℓ, $\ell = 2k+1$, $k > 0$.

1) *Assume* k *odd. Then, there exists a unique 2-fold Pfister form* $\langle\langle a,b\rangle\rangle$ *such that*

   i) $\psi \simeq k\langle\langle a,b\rangle\rangle \perp \langle 1,\det\psi\rangle$

   ii) $D(\langle 1,\det\psi\rangle) = D(\langle\langle a,b\rangle\rangle)$

   iii) $D(\langle\langle a,b\rangle\rangle)$ *is a preordering.*

2) *Assume* $k = 2^r.h$, $r \geqslant 1$, h *odd. Then, there exists a unique* $(r+1)$*-fold Pfister form* ψ_0 *and a unique universal 2-fold Pfister form* ρ *such that*

   i) $\psi \simeq h\psi_1 \perp \langle 1,\det\psi\rangle$ *where* ψ_1 *is a round form defined by* $\psi_0\rho\varphi \simeq \psi_1 \perp 2H$.

   ii) $D(\psi_1)$ *is a preordering.*

   iii) $D(\psi_1) = D(\langle 1,\det\psi\rangle)$.

Proof. 1) By using Prop.2.1 i) we can write

$$\psi \simeq \perp \langle x_i\rangle\varphi_i \perp \langle 1,\det\psi\rangle$$

with: φ_i, 2-fold Pfister form, and $q := \perp \langle x_i\rangle\varphi_i$ a round form of dimension 4k, k odd.

By applying [M], 2.7 (i) it follows the existence of a unique 2-fold Pfister form $\varphi_0 = \langle\langle a,b\rangle\rangle$ such that

$$q \simeq k.\langle\langle a,b\rangle\rangle.$$

Therefore

$$\psi \simeq k\psi_0 \perp \langle 1,\det\psi\rangle.$$

If $k = 1$, then $D(\langle\langle a,b\rangle\rangle) = D(\langle 1,\det\psi\rangle)$ and we know that $D(\langle 1,\det\psi\rangle)$ is a preordering. Assume then $k > 1$. That $k.\langle\langle a,b\rangle\rangle$ is a round form implies, by using [M] 1.7, that $D(\langle\langle a,b\rangle\rangle)$ is a preordering. So $D(\langle\langle a,b\rangle\rangle) = D(k\langle\langle a,b\rangle\rangle) = D(\langle 1,\det\psi\rangle)$.

2) Assume $k = 2^r.h$, $r \geq 1$, h odd. With the notation of Prop.2.1 i) we have that $q = \perp \langle x_i\rangle\varphi_i$ is a round form of dimension $2^{r+1}.h$, h odd. It follows from [M] 2.7, the existence of a unique $\nu+1$-fold Pfister form ψ_0 and a unique universal 2-fold Pfister form ρ such that

$$\rho \simeq h.\varphi_1$$

where φ_1 is defined by

$$\varphi_0 \perp \rho \simeq \varphi_1 \perp 2H.$$

Therefore

$$\psi \simeq h\varphi_1 \perp \langle 1,\det\psi\rangle.$$

By Prop.2.1 we have that

$$D(h.\varphi_1) = D(\langle 1,\det\psi\rangle).$$

If $h = 1$, then φ_1 is round and $D(\varphi_1)$ is a preordering. If $h > 1$, then by [M], 1.7 it follows that $D(\varphi_1)$ is a preordering and φ_1 is round.

2.3. PROPOSITION. *Let* K *be any field and let* ψ *be an anisotropic form over* K*, and* ψ_1 *a round form. Then if* ψ *can be written as*

$$\psi \simeq k.\psi_1 \perp \langle 1,\det\psi\rangle$$

with $k \in N$*,* k *odd, and*

$$D(\psi_1) = D(\langle 1,\det\psi\rangle) \quad \textit{a preordering,}$$

then ψ *is a round form.*

Proof. Since $D(\psi_1)$ is a preordering $\neq$ K, it follows from [M], 1.7 that $k\psi_1$ is a round form if $k > 1$. If $k=1$, the same is clearly true. Let $x \in D(\psi)$, write

$$x = x_1 + x_2,$$
$$x_1 \in D(k\psi_1), \quad x_2 \in D(\langle 1,\det\psi\rangle).$$

Then, $x_1 + x_2 \in D(\langle 1,\det\psi\rangle) = D(\psi_1)$. Therefore $\langle x_1 + x_2\rangle.\psi_1 \simeq \psi_1$, $(x_1 + x_2).k\psi_1 \simeq k\psi_1$ and

$$\langle x_1 + x_2\rangle.\langle 1,\det\psi\rangle \simeq \langle 1,\det\psi\rangle.$$

Consequently

$$\langle x\rangle.\psi = \langle x_1 + x_2\rangle.\psi \simeq \psi,$$

and ψ is round.

2.4. REMARK.

If K is a global field and ψ is an anisotropic round form over K then dim K $\equiv$ 0 (mod 4) and det ψ = 1 (see [HJ]). In fact, since $D(\langle 1,\det\psi\rangle)$ is a preordering, it follows that $\langle 1,\det\psi\rangle$ represents all sum of squares. Therefore, for every discrete prime p in K we have that $\langle 1,\det\psi\rangle_p$ is universal in the completion K_p of K. Now, according to [OM], 63:15 (ii) if φ is a two-dimensional anisotropic form over a local field K and if φ represents 1, then $D(\varphi)$ is a subgroup of K of index 2. Therefore our form $\langle 1,\det\psi\rangle$ is isotropic for all but a finite number of spots (the real ones). Equivalently $-\det\psi$ is a square in all, but a finite number of K_p. By [OM], 65:15, we conclude that $-\det\psi$ is a square in K, i.e. the form $\langle 1,\det\psi\rangle$ is isotropic. This is a contradiction, since it is a subform of an anisotropic form.

After this paper was finished I received a preprint on Round Quadratic Forms by Burkhard Alpers (University of Saskatoon, Canada).

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Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
1428 Nuñez, Buenos Aires, Argentina.

Recibido en febrero de 1989.