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MULTIPLICATIVE INTEGRALS AND GEOMETRY OF SPACES OF PROJECTIONS

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Dedicated to Mischa Cotlar.

INTRODUCTION

Let A be a (real or complex) Banach algebra with identity and $Q_n = \{q = (q_1,\dots,q_n) \in A^n: q_k^2 = q_k,\ q_iq_k = 0 \text{ if } i \neq k,\ \sum_{k=1}^{n} q_k = 1\}$.

For $q_0 = (q_1,\dots,q_n) \in Q_n$ the map $\pi: G \to Q_n$ $\pi(g) = gq_0g^{-1}$ (G denotes the group of units of A) defines a principal fibre bundle over its image (the "joint similarity orbit" of q_0, with the terminology of [CH]); in particular, curves $\gamma: [0,1] \to Q_n$ with origin q_0, admit a lift $\Gamma: [0,1] \to G$, i.e. $\Gamma(t)q_0\Gamma(t)^{-1} = \gamma(t)$ for all $t \in [0,1]$ (see [CPR_1]).

We shall construct, for γ continuous and rectifiable, an explicit lift Γ, which turns to be the horizontal lift of γ for a natural connection on Q_n introduced in [CPR_1]. An analogous result is obtained for the space $S_q = \{(a,b) \in A \times A: aq = a,\ qb = b,\ ba = q\}$ (q is a fixed idempotent of A). Given Banach spaces E and F the set $S(E,F) = \{(i,j) \in L(E,F) \times L(F,E) : ji = 1_E\}$ (considered in [Do] and [Ko]), has the form S_q, when it is non-empty: in fact, if $(i_0,j_0) \in S(E,F)$, $q = i_0j_0$ is an idempotent of $A = L(F)$ and the map $S(E,F) \to S_q$ $(i,j) \mapsto (ij_0,i_0j)$ identifies both spaces. For a fixed $(a_0,b_0) \in S_q$ the map $\pi': G \to S_q$, $\pi'(g) = (ga_0,b_0g^{-1})$ is another principal fibre bundle over its image. As in the case of Q_n a procedure for producing lifts of rectifiable curves $[0,1] \to S_q$, is obtained.

The main advantage of these constructions consists in the fact that they are explicit. They are special cases of the multiplicative integrals of [Po]. The case $n=1$ has been considered in [PR]. The reader is also referred to [CPR_1], [CPR_2] and [CPR_3] for a geometrical study of $Q(=Q_1)$, Q_n and S_q. Earlier facts on Q_n can be found in [Ka_2], [Da], [Ka_1, Ch.II, §4], [DK].

§1. LIFTING RECTIFIABLE CURVES

Let $\gamma: [\alpha,\beta] \to Q_n$ be a continuous rectifiable curve. Our first objective is to construct for every $t \in [\alpha,\beta]$ an invertible element $M_\alpha^t\gamma$ such that

$$(M_\alpha^t\gamma)\gamma(\alpha)(M_\alpha^t\gamma)^{-1} = \gamma(t) \text{ for all } t \in [\alpha,\beta].$$

The idea of the construction is the following: given q,r in Q_n close enough, it is easy to obtain an invertible g such that $gqg^{-1} = r$; now, taking a partition T of $[\alpha,t]$, T: $\alpha = t_0 \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_n = t$ fine enough such that $q^{(i)} = \gamma(t_i)$ is near to $q^{(i+1)}$, we get for each $i = 0,\dots,n-1$ an invertible g_i such that $g_i q^{(i)} g_{i+1} = q^{(i+1)}$; thus $g = g_T = g_n g_{n-1} \cdots g_0 \in G$ conjugates $q^{(0)} = \gamma(\alpha)$ with $q^{(n)} = \gamma(t)$; the difficult task is to prove that $\Gamma(t) = \lim\limits_{\|T\|\to 0} g_T$ exists.

Given $q,r \in Q_n$ we write $L(q,r) = \sum\limits_{k=1}^{n} q_k r_k$.

1.1. PROPOSITION. *For* $q,r,s,q^{(0)},\dots,q^{(m)}$ *in* Q_n *the following properties hold*

(i) $L(q,q) = 1$

(ii) $qL(q,r) = L(q,r)r$

(iii) $L(q,r) - 1 = \sum\limits_{k=1}^{n} q_i(r_i - q_i)$

(iv) $L(q^{(0)},q^{(1)})L(q^{(1)},q^{(2)})\ldots L(q^{(m-1)},q^{(m)}) =$

$$= \sum_{k=1}^{n} q_k^{(0)} q_k^{(1)} \ldots q_k^{(m)}$$

(v) $L(q^{(0)},q^{(1)})\ldots L(q^{(m-1)},q^{(m)})-L(q^{(0)}-q^{(m)}) =$

$$= \sum_{i=1}^{m-1} \sum_{k=1}^{n} q_k^{(0)}(q_k^{(0)}-q_k^{(i)})(q_k^{(i)} - q_k^{(i+1)}) ;$$

(vi) $L(q,r)L(r,s)-L(q,s) = \sum_{k=1}^{n} q_k(q_k-r_k)(r_k-s_k)$.

We omit the proof, which is straightforward; just observe for (v) that, given idempotents $p_0,\ldots,p_m$ in A, by induction on n it follows that

$$p_0p_1\ldots p_{m-1}p_m-p_0p_m = \sum_{i=1}^{m-1} p_0(p_0-p_i)(p_i-p_{i+1}).$$

REMARK. $L(q,r)$ will be the invertible g mentioned at the beginning of this section.

1.2 PROPOSITION. *For each* $K > 0$ *there exists* $K_1 > 0$ *such that, for every* $q,r,s,q^{(0)},\ldots,q^{(m)}$ *in* Q_n *with norm at most* K *it holds that*

(i) $\|q_n\| \leqslant K_1 \quad (k \leqslant n)$

(ii) $\sum_{k=1}^{n} \|q_k-r_k\| \leqslant K_1\|q-r\|$

(iii) $\|L(q,r)\| \leqslant \exp(K_1^2\|q-r\|)$ *and therefore*

$$\|L(q^{(0)},q^{(1)}\ldots L(q^{(m-1)},q^{(m)})\| \leqslant \exp(K_1^2 \sum_{i=1}^{m} \|q^{(i)}-q^{(i-1)}\|)$$

(iv) $\|L(q^{(0)},q^{(1)})\ldots L(q^{(m-1)},q^{(m)})-L(q^{(0)},q^{(m)})\| \leqslant$

$$\leqslant K_1^3 \max_i\|q^{(0)}-q^{(i)}\| \sum_{i=1}^{m} \|q^{(i)}-q^{(i-1)}\|$$

and when m=2,

(v) $\|L(q,r)L(r,s)-L(q,s)\| \leqslant K_1^3\|q-r\| \; \|r-s\|$.

(*The norm we use in* $Q_n \subset A^n$ *is* $\|q\| = \max_{k\leq n}\|q_k\|$).

Proof. Take $K_1 = \max\{n,K\}$. Then, (i) and (ii) follow easily. In order to prove (iii) write $L(q,r) = 1 + \sum_{k=1}^{n} q_k(r_k-q_k)$; using

(i) and (ii)

$$\|L(q,r)\| \leqslant 1 + K_1 \sum_{k=1}^{n} \|r_k - q_k\| \leqslant 1 + K_1^2 \|r-q\| \leqslant \exp(K_1^2 \|q-r\|)$$

Finally, 1.2 (v) implies

$$\|L(q^{(0)},q^{(1)}) \dots L(q^{(m-1)},q^{(m)}) - L(q^{(0)},q^{(m)})\| \leqslant$$

$$\leqslant \sum_{i,k} K_1 \|q_k^{(0)} - q_k^{(i)}\| \|q_k^{(i)} - q_k^{(i+1)}\| \leqslant$$

$$\leqslant K_1^3 \max_i \|q^{(0)} - q^{(i)}\| \sum_i \|q_i - q_{i-1}\| .$$

Suppose that $\gamma: [\alpha,\beta] \to Q_n$ is a (continuous) path and let $K = \max(\|\gamma(t)\|: \alpha \leqslant t \leqslant \beta)$. It follows from 1.2 that there is K_1 such that all conclusions there in hold for elements of the form $\gamma(t)$ with $t \in [\alpha,\beta]$. We will use this repeatedly in the proofs below.

Consider a partition $T: \alpha \leqslant t_0 \leqslant t_1 \leqslant \dots \leqslant t_n = \beta$ and set $\gamma_k = \gamma(t_k)$. We recall that γ has finite length if $\operatorname{Sup} \sum_k \|\gamma_k - \gamma_{k-1}\|$ is finite (Sup taken over all partitions); we denote $\ell(\gamma) = \text{length}(\gamma) = \operatorname{Sup} \sum_k \|\gamma_k - \gamma_{k-1}\|$. More precisely, if $\alpha < u < v < \beta$ we will denote by $\ell(\gamma | [u,v])$ the length of the restriction of γ to $[u,v]$, i.e. the sup taken over all partitions of $[u,v]$.

Given a path $\gamma: [\alpha,\beta] \to Q_n$ and a partition $T: \alpha \leqslant t_0 \leqslant t_1 \leqslant \dots \leqslant t_m \leqslant \beta$ we define

$$M(T) = M(t_0,t_1,\dots,t_m) = L(\gamma_m,\gamma_{m-1}) \dots L(\gamma_1,\gamma_0) .$$

We prove next a series of lemmas that lead to the first theorem of this section whose statement is: $\lim M(T)$ exists for a path of finite length.

We assume from now on that γ has finite length, with $K = \max\|\gamma(t)\|$, $\ell = \text{length}(\gamma)$.

Recall that K_1 is the number of 1.2.; we set $K_2 = \exp(K_1^2)$.

1.3. LEMMA. *For all* T $\|M(T)\| \leq K_2$.

The proof follows easily from 1.1.(iii).

Consider now the partition obtained by adding a new partition point v between t_0 and t_1, i.e. T': $\alpha \leq t_0 \leq v \leq t_1 \leq \ldots \leq \leq t_m \leq \beta$.

1.4. LEMMA. $\|M(T')-M(T)\| \leq K_1^3 K_2 \|\gamma_1-\gamma(v)\| \|\gamma(v)-\gamma_0\|$.

Proof. Write

$$\begin{aligned} M(T')-M(T) &= L(\gamma_m,\gamma_{m-1})\ldots L(\gamma_1,\gamma(v))L(\gamma(v),\gamma_0) - \\ &- L(\gamma_m,\gamma_{m-1})\ldots L(\gamma_1,\gamma_0) = \\ &= M(t_1,t_2,\ldots,t_m)[L(\gamma_1,\gamma(v))L(\gamma(v),\gamma_0)-L(\gamma_1,\gamma_0)] \end{aligned}$$

and therefore, using 1.1 (v) and 1.3

$$\|M(T')-M(T)\| \leq K_2 K_1^3 \|\gamma_1-\gamma(v)\| \|\gamma(v)-\gamma(0)\|$$

as desired.

Suppose now that S: $\alpha \leq s_0 \leq s_1 \leq \ldots \leq s_\ell \leq \beta$ is a partition finer than T: $\alpha \leq t_0 \leq t_1 \leq \ldots \leq t_m \leq \beta$. In order to prove the next lemma we will label the partition points as follows:

$$\alpha \leq s_0 \leq t_0 \leq s_1 \leq \ldots \leq s_h = t_1 \leq s_{h+1} \leq \ldots \leq s_k = t_2 \leq \ldots \text{ etc.}$$

1.5. LEMMA. $\|M(S)-M(T)\| \leq K_1^3 K_2 \; \ell \; \max_j \{\ell(\gamma|[t_j,t_{j+1}])\}$.

Proof. Write

$$\begin{aligned} x_0 &= M(s_0,\ldots,s_h),\ x_1 = M(s_h,\ldots,s_k),\ \text{etc.,} \\ y_0 &= M(t_0,t_1), \\ y_1 &= M(t_1,t_2),\ \text{etc.,} \end{aligned}$$

and

$$w = M(S)-M(T).$$

Then

$$W = x_{m-1}x_{m-2}\dots x_0 - y_{m-1}y_{m-2}\dots y_0 =$$
$$= x_{m-1}\dots x_1(x_0-y_0)+x_{m-1}\dots x_2(x_1-y_1)y_0+ \dots +$$
$$+ x_{m-1}\dots x_{j+1}(x_j-y_j)y_{j-1}\dots y_0+\dots+(x_{m-1}-y_{m-1})y_{m-2}\dots y_0$$

and also $x_{m-1}\dots x_{j+1} = M(s_q,s_{q+1},\dots,s_m)$ with $s_q = t_{j+1}$ and $y_{j-1}\dots y_0 = M(t_0,\dots,t_j)$.

Therefore, using 1.3

$$\|w\| \leqslant K_2^2 \sum_{j=1}^{m} \|x_j-y_j\|.$$

We can write $x_j-y_j = M(u_0,u_1,\dots,u_r)-M(u_0,u_r)$ where $u_0,\dots,u_r$ denote the s_j between t_j and t_{j+1}, so that in particular $u_0 = t_j$, $u_r = t_{j+1}$. Clearly

$$x_j-y_j = M(u_0,\dots,u_r)-M(u_0,u_2,\dots,u_r)+M(u_0,u_2,\dots,u_r) - \dots+M(u_0,u_{r-1},u_r)-M(u_0,u_r),$$

and to each difference we apply 1.4 to obtain

$$\|M(u_0,u_i,u_{i+1},\dots,u_r)-M(u_0,u_{i+1},\dots,u_r)\| \leqslant$$
$$\leqslant K_1^3K_2\|\gamma(u_{i+1})-\gamma(u_i)\|\|\gamma(u_i)-\gamma(u_0)\| \leqslant$$
$$\leqslant K_1^3K_2\ell(\gamma|[t_j,t_{j+1}])\|\gamma(u_{i+1})-\gamma(u_i)\|.$$

Thus $\|x_j-y_j\| \leqslant K_1^3K_2\ell(\gamma|[t_j,t_{j+1}])^2$ and it follows that

$$\|w\| \leqslant K_1^3K_2 \sum_{j=0}^{m-1} \ell(\gamma|[t_j,t_{j+1}]) \leqslant$$
$$\leqslant K_1^3K_2\ \ell\ \max_j \ell(\gamma|[t_j,t_{j+1}]) ,$$

which proves the lemma.

The results of the next lemma are well-known.

1.6. LEMMA. *Let* $\gamma(t)$, $\alpha \leqslant t \leqslant \beta$ *be a continuous path with fini* *te length in a Banach space and let* $\varepsilon > 0$.

Then:

(i) *there exists* $\delta > 0$ *such that if* $\alpha = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_m = \beta$ *is a partition of* $[\alpha,\beta]$ *with* $\max_k |t_k - t_{k-1}| < \delta$ *then* $\ell(\gamma) \leqslant \varepsilon + \sum_1^m \|\gamma(t_k) - \gamma(t_{k-1})\|$;

(ii) *there exists* $\delta' > 0$ *such that if* $[u,v] \subset [\alpha,\beta]$ *and* $0 \leqslant v-u \leqslant \delta'$ *then* $\ell(\gamma|[u,v]) \leqslant \varepsilon$.

1.7. THEOREM. *Let* $\gamma\colon [\alpha,\beta] \to Q_n$ *be a continuous path with finite length. Then, there is an element* $M_\alpha^\beta\gamma$ *when* $\max_k |t_k - t_{k-1}| \to 0$.

Proof. Let $T_1, T_2, \ldots$ be a sequence of partitions of $[\alpha,\beta]$ such that T_{m+1} is finer than T_m and with $|T_m| \to 0$. From 1.5 and 1.6 (ii) it follows that $\{M(T_m)\}$ is a Cauchy sequence in A. Let $M_\alpha^\beta\gamma = \lim M(T_m)$.

Suppose that $\varepsilon > 0$. By 1.6 (ii) there is $\delta > 0$ such that $0 \leqslant v-u \leqslant \delta$ implies $K_1^3 K_2 \ell . \ell(\gamma|[u,v]) \leqslant \varepsilon$. Assume now that W is any partition of $[\alpha,\beta]$ with $|W| \leqslant \delta$. We can write

$$(*) \quad \begin{aligned} \|M_\alpha^\beta\gamma - M(W)\| \leqslant \|M_\alpha^\beta\gamma - M(T_m)\| + \|M(T_m) - M(T_m \cup W)\| \; + \\ + \; \|M(T_m \cup W) - M(W)\| . \end{aligned}$$

From 1.5 with T=W and $S = T_m \cup W$ we get

$$\|M(T_m \cup W) - M(W)\| \leqslant \varepsilon \quad \text{for all } m;$$

again from 1.5 with $T=T_m$ and $S = T_m \cup W$ and using 1.6 we get

$$\lim_m \|M(T_m) - M(T_m \cup W)\| = 0.$$

Hence taking limit for $m \to \infty$ in (*) we obtain $\|M_\alpha^\beta\gamma - M(W)\| \leqslant \varepsilon$ and this concludes the proof.

Clearly, by considering the restrictions of γ to subintervals

$[u,v] \subset [\alpha,\beta]$ we will have elements $M_u^v\gamma$.

1.8. COROLLARY. *If* $f\colon A \to B$ *is a homomorphism of Banach algebras preserving the identities, then* $f(M_\alpha^\beta\gamma) = M_\alpha^\beta(f\circ\gamma)$. *In particular, for every* $g \in G(A)$ $g(M_\alpha^\beta\gamma)g^{-1} = M_\alpha^\beta(g\gamma g^{-1})$.

Proof. It suffices to observe that $f(M_\gamma(T)) = M_{f\circ\gamma}(T)$ for each partition T and apply the theorem.

1.9. PROPOSITION. *For each* $[u,v] \subset [\alpha,\beta]$ *the element* $M_u^v\gamma$ *is invertible in* A *and* $(M_u^v\gamma)^{-1} = M_u^v\sigma$, *where* σ *is the opposite path of* γ, *i.e.,* $\sigma(t) = \gamma(u+v-t)$. *Moreover,* $M_u^v\gamma = 1$ $(u \in [\alpha,\beta])$.

Proof. Let $T\colon u = t_0 \leqslant t_1 \leqslant \dots \leqslant t_m = v$ be a partition of $[u,v]$ and S the "opposite partition", i.e., $s_k = u+v-t_{m-k}$. Then $(M_u^v\gamma)(M_u^v\sigma) = \lim M_\gamma(T).M_\sigma(S)$ as T gets finer, where the notation $M_\gamma(T)$, $M_\sigma(S)$ is self-explanatory.

Clearly $M_\gamma(T)M_\sigma(S) = L(\gamma_m,\gamma_{m-1}).L(\gamma_{m-1},\gamma_{m-2})\dots L(\gamma_1,\gamma_0).L(\sigma_m,\sigma_{m-1}).$ $.L(\sigma_{m-1},\sigma_{m-2})\dots L(\sigma_1,\sigma_0) = L(\gamma_m,\gamma_{m-1})\dots L(\gamma_1,\gamma_0).L(\gamma_0,\gamma_1)\dots L(\gamma_{m-1},\gamma_m)$ where $\gamma_j = \gamma(t_j)$ and $\sigma_k = \sigma(s_k)$. Now, for p,q in P_n it holds $L(p,q)L(q,p) = 1 - \sum_{i=1}^n p_i(q_i-p_i)^2p_i$ so we obtain

$$
\begin{aligned}
M_\gamma(T)M_\sigma(S) &= L(\gamma_m,\gamma_{m-1})\dots L(\gamma_2,\gamma_1)L(\gamma_1,\gamma_0)L(\gamma_0,\gamma_1)\dots L(\gamma_{m-1},\gamma_m) = \\
&= L(\gamma_m,\gamma_{m-1})\dots L(\gamma_2,\gamma_1)\Big(1 - \sum_{i=1}^n \gamma_1^{(i)}(\gamma_0^{(i)}-\gamma_1^{(2)})^2\gamma_1^{(i)}\Big) . \\
&\quad . L(\gamma_1,\gamma_2)\dots L(\gamma_{m-1},\gamma_m) = \\
&= L(\gamma_m,\gamma_{m-1})\dots L(\gamma_2,\gamma_1)\dots L(\gamma_{m-1},\gamma_m) - \\
&\quad - L(\gamma_m,\gamma_{m-1})\dots L(\gamma_2,\gamma_1)\Big(\sum_{i=1}^n (\gamma_0^{(i)}-\gamma_1^{(i)})^2\gamma_1^{(i)}\Big) . \\
&\quad . L(\gamma_1,\gamma_2)\dots L(\gamma_{m-1},\gamma_m).
\end{aligned}
$$

We can apply the same device to the central term $L(\gamma_2,\gamma_1).L(\gamma_1,\gamma_2)$ so we get

$$M_\gamma(T)M_\sigma(S) = L(\gamma_m,\gamma_{m-1})\ldots L(\gamma_3,\gamma_2)L(\gamma_2,\gamma_3)\ldots L(\gamma_{m-1},\gamma_m) -$$

$$- L(\gamma_m,\gamma_{m-1})\ldots L(\gamma_3,\gamma_2)\Big(\sum_{i=1}^{n} \gamma_2^{(i)}(\gamma_1^{(i)}-\gamma_2^{(i)})^2\gamma_2^{(i)}\Big) \,.$$

$$. \, L(\gamma_2,\gamma_3)\ldots L(\gamma_{m-1},\gamma_m) -$$

$$- L(\gamma_m,\gamma_{m-1})\ldots L(\gamma_2,\gamma_1)\Big(\sum_{i=1}^{n} \gamma_1^{(i)}(\gamma_0^{(i)}-\gamma_1^{(i)})^2\gamma_1^{(i)}\Big) \,.$$

$$. \, L(\gamma_1,\gamma_2)\ldots L(\gamma_{m-1},\gamma_m).$$

After m steps we reach, using 1.1 (iv) m times,

$$M_\gamma(T)M_\sigma(S) = 1 - \sum_{k=1}^{n}\sum_{i=0}^{m-1} M_\gamma(T_i)(\gamma_k^{(i)}-\gamma_k^{(i+1)})^2 M_\gamma(S_i)$$

where T_j: $t_j \leqslant t_{j+1} \leqslant \ldots \leqslant t_m$ and

$$S_j\colon s_0 \leqslant s_1 \leqslant \ldots \leqslant s_{n-j-1}.$$

Therefore, taking norms and using 1.3

$$\|M_\gamma(T)M_\sigma(S)-1\| \leqslant K_2^2 \sum_{k=1}^{n}\sum_{i=0}^{m-1} \|\gamma_k^{(i)}-\gamma_k^{(i+1)}\|^2 \leqslant$$

$$\leqslant K_2^2.n \max_i \ell\,(\gamma_i\,|\,[u,v])\ \max_{i,k}\|\gamma_k^{(i)}-\gamma_k^{(i+1)}\|.$$

Taking limits as T gets finer, we conclude from the uniform continuity of γ that

$$\max_{i,k} \|\gamma_k^{(i)}-\gamma_k^{(i+1)}\| \to 0.$$

Thus $\lim M_\gamma(T)M_\sigma(S) = 1$, as claimed.

For $u \leqslant v$ in $[\alpha,\beta]$ we define

$$M_v^u\gamma = (M_u^v\gamma)^{-1}.$$

With this definition we are allowed to write $M_u^v\gamma$ for any pair u,v in $[\alpha,\beta]$.

1.10. PROPOSITION. $(M_u^v\gamma)(M_v^w\gamma) = M_u^w\gamma$ *for all* u,v,w *in* $[\alpha,\beta]$.

Proof. Assume first that $w \leqslant u \leqslant v$. Let T and S be partitions of $[w,u]$ and $[u,v]$, respectively. Clearly $M(S)M(T) = M(T \cup S)$ and taking limits we obtain the desired formula. All other cases reduce to this. For instance, when $w \leqslant v \leqslant u$ we get $M_w^u\gamma = M_w^u\gamma . M_w^v\gamma$, whence $(M_v^u\gamma)^{-1}M_w^u\gamma = M_w^v\gamma$; but $M_u^v\gamma = (M_v^u\gamma)^{-1}$ and the formula follows.

The next result proves the lifting property of $M_v^u\gamma$.

1.11. THEOREM. $(M_u^v\gamma)\gamma(u)(M_u^v\gamma)^{-1} = \gamma(v)$.

Proof. Let T: $t_0 = u \leqslant t_1 \leqslant \dots \leqslant t_w = v$ be a partition of $[u,v]$; then

$$M(T) = L(\gamma_m,\gamma_{m-1})\dots L(\gamma_1,\gamma_0)$$

and each $L(\gamma_j,\gamma_{j-1})$ is invertible if T is fine enough. Then, using 1.1 (ii) m times,

$$M(T)\gamma(u)M(T)^{-1} = M(T)\gamma_0 M(T)^{-1} = \gamma(v)$$

and the theorem follows taking the limit as T gets finer.

If we define, for each t in $[\alpha,\beta]$,

$$\Gamma(t) = M_a^t\gamma \ ,$$

we summarize the properties of $M_u^v\gamma$ as follows.

1.12. THEOREM. (i) Γ *is a function from* $[\alpha,\beta]$ *into* G *with* $\Gamma(\alpha) = 1$; (ii) ("Barrow's rule") $M_u^v\gamma = \Gamma(v)\Gamma(u)^{-1}$ $(u,v \in [\alpha,\beta])$; (iii) $\gamma(t) = \Gamma(t)\gamma(\alpha)\Gamma(t)^{-1}$ $(t \in [\alpha,\beta])$.

Finally, we prove Γ coincides with the lifting of γ

exhibited in [CPR_I], when γ is C^1.

1.13. THEOREM. (i) *If* γ *is a continuous and rectifiable path in* Q_n *then* Γ *is continuous and rectifiable in* G;
(ii) *If* γ *is a* C^1 *path in* Q_n *then* Γ *is also a* C^1 *path in* G *and satisfies* $\dot{\Gamma} = (\sum_{i=1}^{n} \dot{\gamma}_i \gamma_i)\Gamma$, *to* γ.

We need a lemma

1.14. LEMMA. *If* γ *is a path of class* C^1 *and* T: $u = t_0 \leqslant t_1 \leqslant \leqslant \ldots \leqslant t_m = v$ *is a partition, then*

$$\|M(T)-M(u,v)\| \leqslant (v-u)K_2\|\dot{\gamma}\|_\infty \ell(\gamma|[u,v]).$$

Proof. Write

$$M(T)-M(u,v) = M(t_0,\ldots,t_m)-M(t_0,t_m) = $$
$$= \sum_{j=1}^{m-1} M(t_0,t_j,t_{j+1},\ldots,t_m)-M(t_0,t_{j+1},\ldots,t_m).$$

From 1.3 and 1.4 each term can be majorized by

$$K\ell(\gamma|[u,v])\|\gamma_{j+1}-\gamma_j\| \qquad (\text{for } K = K_1^3K_2)$$

so that

$$\|M(T)-M(u,v)\| \leqslant K\ell(\gamma|[u,v]) \sum_{j=0}^{m-1} \|\gamma_{j+1}-\gamma_j\|.$$

The mean value theorem applied to each $\|\gamma_{j+1}-\gamma_j\|$ gives the majorant

$$K\ell(\gamma|[u,v])\|\dot{\gamma}\|_\infty \sum_{j=0}^{m-1} (t_{j+1}-t_j), \text{ as claimed.}$$

Proof of 1.14. Write

$$\Gamma(t+h)-\Gamma(t) = (M_t^{t+h}\gamma)\Gamma(t)-\Gamma(t) = (M_t^{t+h}\gamma-1)\Gamma(t)$$

and define $z = (M_t^{t+h}\gamma-1)-(M(t,t+h)-1)$. Then

$z = \lim(M(t,t_1,\dots,t_{m-1},t+h)-M(t_1,t+h))$.

The inequality used in the proof of 1.14 shows that

$$\|M(t,t_1,\dots,t_{m-1},t+h)-M(t,t+h)\| \leqslant K\ell(\gamma|[t,t+h]) \sum_{j=0}^{m-1} \|\gamma_{j+1}-\gamma_j\| \leqslant$$

$$\leqslant K\ell(\gamma|[t,t+h])^2.$$

By definition of z, $\Gamma(t+h)-\Gamma(t) = (z+M(t,t+h)-1)\Gamma(t)$ so that

$$\|\Gamma(t+h)-\Gamma(t)\| \leqslant \|\Gamma(t)\|(\|z\|+\|M(t,t+h)-1\|) \leqslant$$

$$\leqslant \|\Gamma(t)\|K\ell(\gamma|[t,t+h])^2+\|M(t,t+h)-1\|).$$

Now, $M(t,t+h) = L(\gamma(t+h),\gamma(t))$ and we get, using 1.3,

$$\|M(t,t+h)-1\| \leqslant K'\|\gamma(t)\|\|\gamma(t+h)-\gamma(t)\|.$$

Thus $\|\Gamma(t+h)-\Gamma(t)\| \leqslant (r+s)\|\Gamma(t)\|$, with $r = K\ell(\gamma|[t,t+h])^2$, $s = K'\|\gamma\|_\infty\|\gamma(t+h)-\gamma(t)\|$, which proves the continuity of Γ. Applying the last inequality to each pair $t=t_j$, $t+h = t_{j+1}$ of a partition T of $[\alpha,\beta]$ we get

$$\sum_j \|\Gamma(t_{j+1})-\Gamma(t_j)\| \leqslant \|\Gamma\|_\infty K \sum_j \ell(\gamma|[t_j,t_{j+1}])^2+\|\Gamma\|_\infty K'\ell(\gamma).$$

Taking limits as T gets finer it follows that $\ell(\Gamma) \leqslant \|\Gamma\|_\infty K'\ell(\gamma)$
This proves part (i). Assume now that γ is of class C^1. Then

$$\frac{1}{h}(\Gamma(t+h)-\Gamma(t)) = \frac{1}{h}(M_t^{t+h}\gamma-1)\Gamma(t) ,$$

$$\frac{1}{h}z = \lim_T \frac{1}{h} (M(t,t_1,\dots,t_{m-1},t+h)-M(t,t+h))$$

and by 1.14

$$\left\|\frac{1}{h}z\right\| \leqslant K''\|\dot\gamma\|_\infty\ell(\gamma|[t,t+h])$$

whence $\lim_{h\to 0} \frac{1}{h} z = 0$.

We compute now

$$\lim_{h\to 0}\frac{1}{h}(\Gamma(t+h)-\Gamma(t)) = \lim_{h\to 0}\frac{1}{h}(M_t^{t+h}\gamma-1)\Gamma(t) =$$

$$= \lim_{h\to 0} \left(\frac{1}{h} z + \frac{1}{h}(M(t,t+h)-1)\right)\Gamma(t) =$$

$$= \lim_{h\to 0} \frac{1}{h} \left(L(\gamma(t+h),\gamma(t)-1)\right)\Gamma(t) \ ;$$

but $\frac{1}{h}(L(\gamma(t+h),\gamma(t))-1) = \sum_{i=1}^{n} \gamma_i(t+h) \cdot \frac{1}{h}(\gamma_i(t)-\gamma_i(t+h))$ which converges, when $h \to 0$, to $-\sum_{i=1}^{n} \gamma_i(t)\dot{\gamma}_i(t)$ or, which is the same, to $\sum_{i=1}^{m} \dot{\gamma}_i(t)\gamma_i(t)$. Thus

$$\lim_{h\to 0} \frac{1}{h}(\Gamma(t+h)-\Gamma(t)) = \left(\sum_{i=1}^{n} \dot{\gamma}_i(t)\gamma_i(t)\right)\Gamma(t) \ ,$$

as claimed.

1.15. REMARKS. (i) The results above follow the patterns established in [PR], although the computations here are more envolved.

(ii) The differential equation $\dot{\Gamma} = (\sum\dot{\gamma}_i\gamma_i)\Gamma$ has been found, independently, by Kato [Ka_2] and Daleckii [Da] in the 50's. The geometrical meaning of their solutions, however, has been first established in [CPR_1].

§2. LIFTING CURVES OF BANACH SPACE DECOMPOSITIONS

Recall, from the Introduction, that an idempotent q of a Banach algebra A determines a space

$$S_q = \{(a,b) \in A \times A: aq=a \ , \ qb=b \ , \ ba=q\}.$$

An extensive study of the geometry of S_q, together with some associated maps like $S_q \to Q$, given by $(a,b) \mapsto ab$, can be found in [CPR_4]. A connection ∇ is introduced in S_q and the horizontal lift of a curve in S_q (horizontal with respect to ∇) is obtained by means of a linear differential equation, just as in the case of Q_n (see §1). In this section we descri be the construction of a specific lift of a (rectifiable and continuous) curve γ in S_q, which turns to be, when γ is, say, of class C^1, the horizontal lift of γ. We shall follow the lines of §1 so that we only need to prove the main inequalities.

Given $\delta = (a,b)$, $\delta' = (a',b')$ in S_q consider $L(\delta,\delta') = ab' + (1-r)(1-r')$, where $r = ab$ and $r' = a'b'$; observe that both are idempotent elements of A.

It is clear that, for δ,δ' close enough, $L(\delta,\delta')$ is invertible. Moreover, if we define an action of G over S_q by $g.\delta = (ga,bg^{-1})$, it is clear that $L(\delta,\delta').\delta' = \delta$.

2.1. PROPOSITION. *Let* δ,δ',δ'' *be fixed elements of* S_q *with norm at most* K. *There exists* $K_1 > 0$ *such that*

(i) $\|L(\delta,\delta')\| \leq 1 + K_1\|\delta-\delta'\|$

(ii) $\|L(\delta,\delta')L(\delta',\delta'')-L(\delta,\delta'')\| \leq K_1\|\delta-\delta'\|\|\delta'-\delta''\|$.

Proof. Both inequalities follow easily from the identities

(2.1.1) $L(\delta,\delta') = 1 + (1-r)(a-a')b' + a(b'-b)$

(2.1.2) $L(\delta,\delta')L(\delta',\delta'')-L(\delta,\delta'') =$

$= [(a'-a)b' + ab(a-a')b'][a'(b'-b'') + a'b'(a'-a'')b'']$.

Let $\gamma: [\alpha,\beta] \to S_q$ be a rectifiable continuous curve and consider a partition of $[\alpha,t]$, $T: t_0 = \alpha \leq t_1 \leq \ldots \leq t_n = t$ fine enough such that, for every $k = 1,2,\ldots,n$, $g_k = L(\gamma(t_k),\gamma(t_{k-1}))$ is invertible. Denoting $\gamma(t) = (a(t),b(t))$ and $\gamma(t_k) = \gamma_k = (a_k,b_k)$, we shall prove that all invertible elements of the form $g_T = g_n g_{n-1}\ldots g_1$ are uniformly bounded, independently of T.

2.2. LEMMA. *There is a constant* M, *depending only on* γ, *such that for every partition* T *of* $[\alpha,t]$ $\|g_T\| \leq M$.

Proof. By 2.1 (i), with $K = \sup\|\gamma(t)\|$, it follows that

$\|g_k\| \leqslant 1 + K_1\|\gamma_k-\gamma_{k-1}\|$ so that

$$\|g_T\| \leqslant \prod_{k=1}^{n} \|g_k\| \leqslant \prod_{k=1}^{n} (1 + K_1\|\gamma_k-\gamma_{k-1}\|) \leqslant$$

$$\leqslant \exp \sum_{k=1}^{n} K_1\|\gamma_k-\gamma_{k-1}\| \leqslant \exp(K_1\ell_t) = M.$$

where ℓ_t is the length of $\gamma|[\alpha,t]$.

2.3. LEMMA. *Given two partitions*

$$T_1: s \leqslant w \leqslant v_0 \leqslant \ldots \leqslant v_n \quad ; \quad T_2: s \leqslant v_0 \leqslant \ldots \leqslant v_n$$

of $[s,v_n] \subset [\alpha,\beta]$, *for the same constant* K_1 *of* 2.1, *it holds that*

$$\|g_{T_1}-g_{T_2}\| \leqslant K_1\|g_{T'}\|\ell_s^{v_n}\|\gamma(v_0)-\gamma(w)\|$$

where $T': v_0 \leqslant v_1 \leqslant \ldots \leqslant v_n$ *and* $\ell_s^{v_n}$ *is the length of* $\gamma|[s,v_n]$.

Proof. Let $x = L(\gamma(v_0),\gamma(w))L(\gamma(w),\gamma(s))-L(\gamma(v_0),\gamma(t))$. By 2.1 (ii)

$$\|x\| \leqslant K_1\|\gamma(w)-\gamma(s)\|\|\gamma(w)-\gamma(v_0)\| \leqslant K_1\|\gamma(w)-\gamma(v_0)\|\ell_s^{v_n}.$$

Let $z = g_{T_1}-g_{T_2} = g_{T'}x$. Then

$$\|z\| \leqslant \|g_{T'}\|\|x\| \leqslant K_1\|g_{T'}\|\ell_s^{v_n}\|\gamma(v_0)-\gamma(w)\| \quad .$$

2.4. LEMMA. *Given partitions*

$$S = (s_0,\ldots,s_m) \quad , \quad T = (t_0,\ldots,t_n) \text{ of } [\alpha,t],$$

S *finer than* T, *there exists a constant* K_2 *depending only on* γ *such that*

$$\|g_S-g_T\| \leqslant K_2\ \ell_\beta \max_k \ell_{t_k}^{t_{k+1}} \quad .$$

We omit the proof because is a simple variation of that of 1.5, using the estimates of 2.2 and 2.3. Moreover the rest of the proof of the following theorem follows the lines of 1.7-1.14, just replacing the corresponding lemmas by these proven in this section.

2.5. THEOREM. *Let* γ: $[\alpha,\beta] \to S_q$ *be a rectifiable and continuous curve in* S_q. *Then, for every* $t \in [\alpha,\beta]$ *the limit* $\Gamma(t) = \lim_{\|T\|\to 0} g_T$ *exists and* Γ *is a rectifiable and continuous curve* $[\alpha,\beta] \to G$ *such that* $\Gamma(t).\gamma(\alpha) = \gamma(t)$ *for all* $t \in [\alpha,\beta]$. *If* γ *is a* C^1*-curve then* Γ *is the unique solution of the initial value problem*

$$\dot{\Gamma} = (-a\dot{b} + \dot{r}r)\Gamma \ , \quad \Gamma(\alpha) = 1.$$

Proof. We only consider the last assertion. Let $t \in [\alpha,\beta]$, $h > 0$ and T a partition of $[\alpha,t]$ T: $t_0 = \alpha < t_1 < \ldots < t_n = t$. If h is small enough and $g_{t+h} = a(t+h)b(t)+(1-r(t+h))(1-r(t))$,

$$\Gamma(t+h)-\Gamma(t) = g_{t+h}\Gamma(t)-\Gamma(t)+o(h),$$

by the first part of the theorem; so it is enough to prove that $\lim_{h\to 0} (g_{t+h}-1)/h = -a(t)\dot{b}(t)+\dot{r}(t)r(t)$.

Computing

$$\begin{aligned} g_{t+h}-1 &= a(t+h)b(t)-r(t+h)-r(t)+r(t+h)r(t) = \\ &= a(t+h)(b(t)-b(t+h))+(r(t+h)-r(t)r(t)) \end{aligned}$$

and the result follows.

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