Revista de la
Unión Matemática Argentina
Volumen 34, 1988.

ON A GEOMETRIC INTERPRETATION OF SCHUR PARAMETERS

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Para Mischa Cotlar:
a la Matemática le debo
un maestro de excepción
y un amigo muy querido.

ABSTRACT. As an essentially self-contained introduction to a general approach to moment type problems, based on an original idea due to Mischa Cotlar, we sketch a method to solve the classical Caratheodory-Fejer problem and give a geometric interpretation of the Schur parameters.

INTRODUCTION

In the late seventies, Cotlar suggested that a class of singular integrals on weighted spaces could be studied by means of a modification of Toeplitz kernels, an idea that was first applied through the Cotlar-Sadosky lifting theorem [C-S.1]. That kind of kernels was later included in the notion of "Generalized Toeplitz Kernels" [A-C], which allows a unified approach to several problems (see [C-S.2] for a general overview).

By means of a further generalization of that notion, the *Toeplitz-Krein-Cotlar forms* [Ar.1], [Ar.2], such subjects as the extension to the discrete plane of a theorem of Krein, on the Fourier transform of a function of positive type on an interval, and of the Nagy-Foias lifting of the commutant can be considered in the same framework. The basic idea is that several gener alized moment problems give rise to a family of isometric operators, with domains and ranges depending on the operator, such that the original problem can be solved iff there exists a family of commuting unitary extensions of those operators. In this way not only existence questions can be handled; also unicity conditions and descriptions of all the solutions in the indeterminate case appear in quite a natural way. In particular, a simple geometric interpretation can be given in this framework of the "choice sequences" introduced in [C-F] and [A-C-F] as a far reaching extension of the Schur parameters.

Now, those parameters, nowadays so important in several subjects (see [K]), were introduced by Schur [Sch.] as an analytic tool to deal with the classical Caratheodory-Fejer problem. So, as a hopefully simple introduction to the general method we summarized before, we want to show in this paper how the above mentioned Cotlar's idea leads to an operator-theoretic solution of that problem and to a geometric characterization of Schur parameters.

THE CARATHEODORY-FEJER PROBLEM REVISITED

NOTATION. Let $\mathbf{T}$ be the unit circle in the complex plane C, m the normalized Lebesgue measure in $\mathbf{T}$ and $\mathbf{Z}$ the set of integral numbers. If $f \in L^1(\mathbf{T}) \equiv L^1(\mathbf{T},m)$, its Fourier transform $\hat{f}: \mathbf{Z} \to C$ is given by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-ikx}\,dx \equiv \int_{\mathbf{T}} f(z)\bar{z}^k\,dm(z)$$

The support of any function h is the set supp h $:= \{h \neq 0\}$; $f \in L^1(\mathbf{T})$ is a trigonometric polynomial if supp $\hat{f}$ is a finite set. For $1 \leqslant p \leqslant \infty$, $H^p(\mathbf{T})$ is the Hardy space defined by

$$H^p(\mathbf{T}) = \{f \in L^p(\mathbf{T}) : \hat{f}(k) = 0 \text{ if } k < 0\}.$$

THE PROBLEM. *If a sequence* $\{c_0, c_1, \ldots, c_n, \ldots\} \subset C$ *is given* *and* F_n *denotes the set of all the functions* $f \in H^{\infty}(\mathbf{T})$ *such that*

$$\hat{f}(k) = c_k, \quad k = 0,1,\dots,n \ ; \ \|f\|_{\infty} \leqslant 1 \ ,$$

the problem is to characterize, for each natural n*, the* (n+1)-*uples such that* F_n *is non void.*

A NECESSARY CONDITION.

Assume that there exists $f \in F_n$; then the matrix $(a_{ij})_{i,j=1,2}$ given by $a_{11} = a_{22} = 1$, $a_{12} = f$, $a_{21} = \bar{f}$, is positive semidefinite a.e. in $\mathbf{T}$. Thus

$$0 \leqslant \int_{\mathbf{T}} [|g_1|^2 + \bar{f}g_2\bar{g}_1 + fg_1\bar{g}_2 + |g_2|^2] \ dm$$

holds for any trigonometric polynomials g_1, g_2. It follows that

$$(1) \ \ 0 \leqslant \Sigma\{\delta(u-v)h_1(u)\bar{h}_1(v) + 2\mathrm{Re}[\hat{f}(u-v)h_1(u)\bar{h}_2(v)] + \delta(u-v)h_2(u)\bar{h}_2(v) : u,v \in \mathbf{Z}\}$$

is true for any $h_1, h_2 \colon \mathbf{Z} \to C$ with finite support, where $\delta \colon \mathbf{Z} \to C$ is such that $\operatorname{supp} \delta = \{0\}$ and $\delta(0) = 1$.

It is easy to see that, if we set $c_k = 0$ for every $k < 0$ and $W \equiv W(n) := \{k \in \mathbf{Z} \colon 0 \leqslant k \leqslant n\}$, then (1) is equivalent to

$$(2) \ \ 0 \leqslant \Sigma\{|h_1(u)|^2 : u \in W\} + 2\mathrm{Re}\Sigma\{c_{u-v}h_1(u)\bar{h}_2(v) : u,v \in W\} + \Sigma\{|h_2(v)|^2 : v \in W\},$$

$\forall \ h_1, h_2 \colon \mathbf{Z} \to C$ such that $\operatorname{supp} h_1$, $\operatorname{supp} h_2 \subset W$.

Now, the last condition depends only on the given data $\{c_0, c_1, \dots, c_n\}$ and is the same as saying that the operator Γ_n on C^{n+1} given by the Toeplitz matrix $(t_{uv})_{0 \le u,v \le n}$, with $t_{uv} = c_{v-u}$, satisfies $\|\Gamma_n\| \leqslant 1$. Summing up:

For F_n *to be non void it is necessary that* $\|\Gamma_n\| \leqslant 1$.

AN AUXILIARY FORM.

In order to prove that the above necessary condition is also sufficient, we consider (1) as the assertion that a Toeplitz form constructed by means of f and acting in the space

$$A := \{h = (h_1,h_2),\ h_1,h_2\colon \mathbf{Z} \to C \text{ with finite support}\}$$

is positive, and we observe that knowing $\{c_0,c_1,\dots,c_n\}$ is the same as knowing the restriction of that form to a well-defined subspace of A. These remarks motivate the following construction.

Set $W_1 = W_1(n) = \{k \in \mathbf{Z}\colon k \leq n\}$, $W_2 = \{k \in \mathbf{Z}\colon 0 \leq k\}$,

$$A(n) = \{h = (h_1,h_2) \in A\colon \text{supp } h_1 \subset W_1(n),\ \text{supp } h_2 \subset W_2\}$$

and define a form B: $A(n) \times A(n) \to C$ setting, for any $h,h' \in A(n)$,

$$\text{(3)}\quad B(h,h') = \Sigma\{h_1(u)\bar{h}_1'(u)\colon u \in W_1\} + \Sigma\{c_{u-v}h_1(u)\bar{h}_2'(v)\colon (u,v) \in W_1{\times}W_2\} + \\ + \Sigma\{\bar{c}_{u-v}h_2(v)\bar{h}_1'(u)\colon (u,v) \in W_1{\times}W_2\} + \Sigma\{h_2(v)\bar{h}_2'(v)\colon v \in W_2\}.$$

Clearly, B is a sesquilinear form; it is easy to see that $\|\Gamma_n\| \leq 1$ implies that B is positive, i.e., such that $B(h,h) \geq 0$, $\forall\ h \in A(n)$. Now, B is an example of a generalization of the classical notion of a Toeplitz form in the following sense: let S be the shift, i.e., $Sg(m) \equiv g(m-1)$ for every g in A; then a Toeplitz form in A is an S-invariant form, while it is not difficult to prove that

$$\text{(4)}\qquad B(Sh,Sh') = B(h,h')$$

whenever it makes sense, that is, for every h, h' in $D'(n) := \{g \in A(n)\colon Sg \in A(n)\}$.

Now we proceed as in the proof of the famous Naimark's dilation theorem (see [N-F]): setting $\langle h,h'\rangle = B(h,h')$ for every $h,h' \in A(n)$, the positive form B and the vector space A(n) generate a Hilbert space and a canonic map Λ from A(n) onto a dense subspace of H(n), while $S_{|D'(n)}$ defines in the natural way (i.e., $V\,\Lambda_{|D'(n)} = \Lambda\, S_{|D'(n)}$) an isometric operator V from D(n) onto R(n), which are both subspaces of H(n).

If follows from that construction that $d_1 := \Lambda(\delta,0)$, $d_2 := \Lambda(0,\delta)$ implies

(5) $$c_k = \langle V^k d_1, d_2\rangle \ , \ 0 \leqslant k \leqslant n \ ,$$

so it is natural to try to define c_k for $k > n$ by extending V in such a way that (5) still makes sense. In fact, it is not difficult to see that there exists a Hilbert space G containing H(n) and a unitary operator $U \in \mathbf{L}(G)$ that extends V. Let E be the spectral measure of U^{-1}; we define a positive matrix of Borel measures in $\mathbf{T}$, $M = (M_{ij})_{i,j=1,2}$, setting $M_{ij}(.) = \langle E(.)d_i, d_j\rangle_G$ and we calculate the Fourier coefficients of these measures:

$$\hat{M}_{ij}(k) := \int_{\mathbf{T}} z^{-k} dM_{ij}(z) = \int_{\mathbf{T}} z^{-k} d\langle E(z)d_i, d_j\rangle_G = \langle U^k d_i, d_j\rangle_G,\ k \in \mathbf{Z},\ i,j=1,2.$$

For $k \geqslant 0$ we have $\langle U^{-k}d_1, d_1\rangle_G = \langle V^{-k}d_1, d_1\rangle_{H(n)} = B[S^{-k}(\delta,0), (\delta,0)] = \delta(-k)$, as it follows from the definition of B. Thus, the real measure M_{11} is simply Lebesgue measure m, and the same holds for M_{22}. Since M is a positive matrix, M_{12} has to be absolutely continuous with respect to m, i.e., $dM_{12} = f\, dm$, with $f \in L^1(\mathbf{T})$. Thus the matrix $(a_{ij})_{i,j=1,2}$ given by $a_{11} = a_{22} = 1$, $a_{12} = f$, $a_{21} = \bar{f}$, is positive semidefinite a.e so $\|f\|_\infty \leqslant 1$. Moreover, for any k we have

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-ikx}dx = \hat{M}_{12}(-k) = \langle U^k d_1, d_2\rangle_G \ ,$$

so $\hat{f}(k) = \langle V^k d_1, d_2\rangle_n = B[S^k(\delta,0),(0,\delta)] = c_k$ holds for every $k \leqslant n$.

So the proof that $F_n \neq \phi$ iff $\|\Gamma_n\| \leqslant 1$ is over.

DESCRIPTION OF ALL SOLUTIONS.

Let $\mathbf{U^*}$ be the set of all the (U,G) such that G is a Hilbert space containing H(n) and $U \in \mathbf{L}(G)$ a unitary operator that extends V. To each $(U,G) \in \mathbf{U^*}$ we associate $f \in F_n$ characterized by

(6) $$\hat{f}(k) = \langle U^k d_1, d_2 \rangle_G \ , \ k \in \mathbf{Z}.$$

The function $f \in H^\infty(\mathbf{T})$ is obtained as the boundary value of an analytic function in $\mathbf{D} = \{z \in C: |z| < 1\}$, which we also call f and is given by

$$f(z) = \Sigma\{\hat{f}(k) z^k: \ k \in \mathbf{Z}\} \equiv \Sigma\{\hat{f}(k) z^k: \ k \geqslant 0\}$$

so the correspondence $(U,G) \to f$ is given by

(7) $$f(z) = \langle (I - zU)^{-1} d_1, d_2 \rangle_G \ , \ |z| < 1.$$

In order to see that this correspondence from $\mathbf{U^*}$ to F_n is surjective, remark that, if $f \in F_n$ is given and $c_u = \hat{f}(u)$ for every $u \in \mathbf{Z}$, then (3) defines a positive S-invariant form $B: A \times A \to C$, so as before A and B generate a Hilbert space G while S generates a unitary operator $U \in \mathbf{L}(G)$ that extends V and such that (6) holds, so f is given by $(U,G) \in \mathbf{U^*}$.

Consequently, all the solutions of the Caratheodory-Fejer problem can be obtained by the method we have sketched.

Moreover, the $(U,G) \in \mathbf{U^*}$ we have just obtained from a given $f \in F_n$ satisfies also the minimality condition

(8) $$G = \bigvee\{U^n H(n): \ n \in \mathbf{Z}\} \ ,$$

where $\bigvee\{\ldots\}$ denotes as usual the smallest Hilbert space that contains all the sets in $\{\ldots\}$, so we say that such (U,G) is a *minimal unitary extension* of V. Consequently, in order to get the functions of F_n we can restrict the above considered correspondence to

$$\mathbf{U} := \{(U,G) \in \mathbf{U^*}: U \text{ is a minimal extension of } V\}$$

Note that, for any $(U,G) \in \mathbf{U}$, $\langle U^k d_j, d_j \rangle_G = 0$ if $k \neq 0$, $j=1,2$; in fact, if $j=1$ and $k < 0$ then $0 = B[S^k(\delta,0),(\delta,0)] =$
$= \langle \Lambda S^k(\delta,0), \Lambda(\delta,0) \rangle_{H(n)} = \langle V^k d_1, d_1 \rangle_n = \langle U^k d_1, d_1 \rangle_G$, etc.

If $(U',G'),(U'',G'') \in \mathbf{U}$ correspond to the same $f \in F_n$ then

$$\langle U'^k d_i, U'^m d_j \rangle_{G'} = \langle U''^k d_i, U''^m d_j \rangle_{G''}$$

holds for any $k,m \in \mathbf{Z}$, $i,j = 1,2$, so setting $\tau(U'^k d_i) = U''^k d_i$ we define a unitary operator τ from G' onto G" such that

$$\tau U' = U''\tau \quad , \quad \tau_{H(n)} = I_{H(n)}$$

That is, U' y U" are essentially the same extension of V, so we write $(U',G') \approx (U'',G'')$ and we consider that they are equal as elements of **U**.

With that identification, the correspondence between **U** and F_n is bijective: $\mathbf{U} \leftrightarrow F_n$.

ON THE UNICITY OF THE SOLUTION.

F_n will have only one element when the same happens with **U**. Let D(n) be the domain of V, R(n) its range and N(n), M(n) its defect subspaces, i.e., the orthogonal complements in H(n) of D(n), R(n), respectively. It is not difficult to see that V has essentially only one minimal unitary extension if at least one of its defect subspaces (which in this case have the same dimension) is trivial.

Now, D(n) = H(n) iff $\Lambda S^n(\delta,0) = V^n d_1 \in D(n)$, so we have to study

$$\rho(n) := \mathrm{dist}^2[V^n d_1, \Lambda D'(n)]$$

which is the infimum, for $h = (h_1,h_2) \in A(n)$ with $h_1(n) = 0$, of $B[S^n(\delta,0)+h, S^n(\delta,0)+h] = 1 + \Sigma\{|h_1(u)|^2 : u < n\} +$ $+ 2\mathrm{Re}\Sigma\{c_{n-v}\bar{h}_2(v) : v \geq 0\} + 2\mathrm{Re}\Sigma\{c_{u-v}h_1(u)\bar{h}_2(v) : v \geq 0,\ u < n\} +$ $+ \Sigma\{|h_2(v)|^2 : v \geq 0\}$. Thus, with obvious notation:

$\rho(n) = \inf\{\|h_1\|^2 + \|h_2\|^2 + 2\mathrm{Re}\langle \Gamma_n h_1, h_2\rangle : h_1,h_2 \in C^{n+1},\ h_1(n) = 1\}$.

Replacing h_2 (when $\Gamma_n h_1$ is not zero) by $-(\|h_2\|/\|\Gamma_n h_1\|)\Gamma h_1$ we can see that $\rho(n) = \inf\{\|h\|^2 - \|\Gamma_n h\|^2 : h \in C^{n+1},\ h(n) = 1\}$.

Clearly, D(n) = H(n) iff $\rho(n) = 0$, and R(n) = H(n) iff $\rho'(n)=0$, with $\rho'(n) = \inf\{\|h\|^2 - \|\Gamma_n^* h\|^2 : h \in C^{n+1},\ h(0) = 1\}$. Now, if J is the antilinear transformation in C^{n+1} given by

$(Jh)(j) = \bar{h}(n-j)$, $0 \leqslant j \leqslant n$, then $J\Gamma_n = \Gamma_n{}^*J$, so $\{\|h\|^2-\|\Gamma_n{}^*h\|^2: h \in C^{n+1}, h(0) = 1\} = \{\|Jh\|^2-\|\Gamma_n{}^*Jh\|^2: h \in C^{n+1}, h(n) = 1\} = \{\|Jh\|^2 - \|J\Gamma_n h\|^2 : h \in C^{n+1}, h(n) = 1\}$; thus, $\rho(n) = \rho'(n)$.

Since $\Gamma_n{}^*\delta = (\bar{c}_0,\ldots,\bar{c}_n)$, $\rho(n) \leqslant \|\delta\|^2 - \|\Gamma_n{}^*\delta\|^2 = 1 - \Sigma\{|c_j|^2 : 0 \leqslant j \leqslant n\}$.

If $\|\Gamma_n\| \leqslant 1$ and $h(n) = 1$, then $\|h\|^2 - \|\Gamma_n h\|^2 \geqslant \|h\| - \|\Gamma_n h\| \geqslant 1 - \|\Gamma_n\|$, so $\|\Gamma_n\| \leqslant 1$ implies $\rho(n) \geqslant 1 - \|\Gamma_n\|$. In this way we arrive at the following

(9) PROPOSITION. *Given* $\{c_0,c_1,\ldots,c_n\} \subset C$, *set* $F_n = \{f \in H^\infty(\mathbf{T}) : \hat{f}(k) = c_k,\ k=0,1,\ldots,n;\ \|f\|_\infty \leqslant 1\}$, *let* Γ_n *be the operator in* C^{n+1} *given by the matrix* $(t_{uv})_{0\leq u,v\leq n}$, *with* $t_{uv} = c_{v-u}$ *if* $u \leqslant v$ *and* $t_{uv} = 0$ *if* $u > v$, *and set* $\rho(n) = \inf\{\|h\|^2 - \|\Gamma_n h\|^2 : h \in C^{n+1}, h(n) = 1\}$. *Then*:

a) $F_n \neq \phi \Leftrightarrow \|\Gamma_n\| \leqslant 1 \Leftrightarrow \rho(n) \geqslant 0 \Rightarrow \rho(n) \geqslant 1 - \|\Gamma_n\|$.

b) $\rho(n) = \inf\{\|h\|^2 - \|\Gamma_n{}^*h\|^2 : h \in C^{n+1}, h(0) = 1\} \leqslant 1-\Sigma\{|c_j|^2 : 0 \leqslant j \leqslant n\}$

c) $\#(F_n) = 1 \Leftrightarrow \rho(n) = 0$.

It only remains to prove that $\rho(n) \geqslant 0$ implies $\|\Gamma_n\| \leqslant 1$. Assume that $\|\Gamma_n\| > 1$. Let $h' \in C^{n+1}$ be such that $\|h'\| = 1$ and $\|\Gamma_n h'\| = a > 1$. If h' does not belong to $C^n = \{g \in C^{n+1} : g(n) = 0\}$ there exists a non zero scalar b such that $bh'(n) = 1$; if $h = bh'$ then $\rho(n) \leqslant \|h\|^2 - \|\Gamma_n h\|^2 = |b|^2(1-a^2) < 0$. If $h' \in C^n$, set $h = bh' + v$, b scalar and $v = (0,\ldots,1)$; then $h(n) = 1$ and $\|h\|^2 - \|\Gamma_n h\|^2 = |b|^2\|h'\|^2 + 1 - |b|^2\|\Gamma_n h'\|^2 - \|\Gamma_n v\|^2 - 2\mathrm{Re}[b\langle\Gamma_n h',\Gamma_n v\rangle] = |b|^2(1-a^2) - 2\mathrm{Re}[b\langle\Gamma_n h',\Gamma_n v\rangle] + 1 - \Sigma\{|c_j|^2 : 0 \leqslant j \leqslant n\}$, which is negative for a convenient b; thus $\rho(n) < 0$.

CONDITIONS FOR THE EXISTENCE AND UNICITY OF SOLUTIONS.

$\|\Gamma_n\| \leqslant 1$ iff the operator $(I - \Gamma_n{*}\Gamma_n)$ is positive semidefinite. If an operator B is given by the matrix $(b_{uv})_{0\leq u,v\leq n}$ and if Δ_m denotes the determinant of $(b_{uv})_{0\leq u,v\leq m}$, then: i) B is positive semidefinite (i.e., $\langle Bh,h\rangle \geqslant 0$ for every h) iff $\Delta_m \geqslant 0$ for $0 \leqslant m \leqslant n$; ii) B is positive definite (i.e., $\langle Bh,h\rangle > 0$ for every non zero h) iff $\Delta_m > 0$ for $0 \leqslant m \leqslant n$. Now:

(10) $$\rho(n) = 0 \iff \|\Gamma_n\| = 1 \iff$$
$$(I - \Gamma_n{*}\Gamma_n) \text{ is positive semidefinite and } \det(I - \Gamma_n{*}\Gamma_n) = 0$$

It is easy to prove the equivalence between the second and the third condition. Let $\|\Gamma_n\| \leqslant 1 \iff \rho(n) \geqslant 0$.
If $\det(I - \Gamma_n{*}\Gamma_n) = 0$, there exists a non zero $h \in C^{n+1}$ such that $(I - \Gamma_n{*}\Gamma_n)h = 0$, so $\|h\|^2 = \|\Gamma_n h\|^2$. Let $m \leqslant n$ maximum such that $h(m) \neq 0$ and assume $h(m) = 1$. From $\|h\|^2 = \|\Gamma_m h\|^2$ we get $\rho(m) = 0$, so $\#(F_m) = 1$ and $\rho(n) = 0$.

Conversely, if $\rho(n) = 0$, there exists $\{h_v: v \geqslant 0\} \subset C^{n+1}$ such that $h_v(n) \equiv 1$ and $\|h_v\|^2 - \|\Gamma_n h_v\|^2$ goes to 0. If $\{h_v\}$ has a bounded subsequence, $\rho(n)$ is a minimum, so $\|\Gamma_n\| = 1$. If $\|h_v\|$ goes to ∞, we can find a vector g such that $\|g\| = 1$ and $\|g\|^2 - \|\Gamma_n g\|^2 = 0$, etc.
The proof of (10) is over.
We now show how ρ can be calculated when $\det(I - \Gamma_n{*}\Gamma_n) > 0$. Let $B = (I - \Gamma_n{*}\Gamma_n)$ be given by the matrix $(b_{uv})_{0\leq u,v\leq n}$ with respect to the canonic base $\{e_0,e_1,\ldots,e_n\}$ in C^{n+1} and call Δ_m the determinant of the matrix $(b_{uv})_{0\leq u,v\leq m}$. Orthonormalizing $\{e_0,e_1,\ldots,e_n\}$ with respect to the scalar product defined by the positive operator B, i.e., $(h,h') := \langle Bh,h'\rangle$, we obtain

a basis $\{g_0, g_1, \ldots, g_n\}$ such that also $g_m(m) = \sqrt{(\Delta_{m-1}/\Delta_m)}$ if $m > 0$ and $g_0(0) = \sqrt{\Delta_0^{-1}}$. Given $h = \Sigma\{a_j g_j : 0 \leqslant j \leqslant n\}$ then $h(n) = a_n g_n(n)$, so $h(n) = 1$ iff $a_n = 1/g_n(n)$; moreover, $\langle Bh,h\rangle = \Sigma\{|a_j|^2 : 0 \leqslant j \leqslant n\}$. Thus, $\rho(n) = \inf\{\langle Bh,h\rangle : h(n)=1\} = 1/|g_n(n)|^2$. Consequently:

(11) Let $\rho(n) > 0 \iff (I-\Gamma_n{}^*\Gamma_n)$ is positive definite $\iff \|\Gamma_n\| < 1$; if Δ_m, $0 \leqslant m \leqslant n$, are the principal minors of the matrix $(I-\Gamma_n{}^*\Gamma_n)$ and $\Delta_{-1} = 1$ then $\rho(n) = \Delta_n/\Delta_{n-1}$.

Now, when $\rho(n)$ is positive, it is not difficult to prove that the isometry V has an infinite number of essentially different minimal unitary extensions, so F is infinite. Thus, a proof has been given of the following:

THEOREM A. *Given* $\{c_0, c_1, \ldots, c_n\} \in C^{n+1}$, *set*

$$F_n = \{f \in H^\infty(\mathbf{T}) : \hat{f}(k) = c_k,\ k = 0,1,\ldots,n\ ;\ \|f\|_\infty \leqslant 1\}.$$

a) *Let* Γ_n *be the operator in* C^{n+1} *whose matrix with respect to the canonic base is* $(t_{uv})_{0\leq u,v\leq n}$ *with* $t_{uv} = c_{v-u}$ *if* $u \leqslant v$ *and* $t_{uv} = 0$ *if* $u > v$; *set* $\rho(n) = \inf\{\|h\|^2 - \|\Gamma_n h\|^2 : h \in C^{n+1},\ h(n) = 1\}$. *Then:* F_n *is non empty* $\iff \|\Gamma_n\| \leqslant 1 \iff I - \Gamma_n{}^*\Gamma_n$ *is positive semidefinite* $\iff \rho(n) \geqslant 0$.

b) F_n *has only one element* $\iff \|\Gamma_n\| = 1 \iff I - \Gamma_n{}^*\Gamma_n$ *is positive semidefinite and* $\det(I - \Gamma_n{}^*\Gamma_n) = 0 \iff \rho(n) = 0$.

c) F_n *has more that one element* $\iff \#(F_n) = \infty \iff \|\Gamma_n\| < 1 \iff I - \Gamma_n{}^*\Gamma_n$ *is positive definite* $\iff \rho(n) > 0$.

d) *When* $(I - \Gamma_n{}^*\Gamma_n)$ *is positive definite, if* Δ_m, $0 \leqslant m \leqslant n$, *are the principal minors of the matrix* $(I - \Gamma_n{}^*\Gamma_n)$ *and* $\Delta_{-1} = 1$, *then* $\rho(n) = \Delta_n/\Delta_{n-1}$.

For classical proofs and corresponding references see [Ak.].

A recent account on the relations between operator theory and moment problems is given in [Sa].

A CHARACTERIZATION OF SCHUR PARAMETERS.

For each n we have a space H(n) and an isometry V with domain D(n) and range R(n). In fact, that operator should be called V(n), but since the restriction of V(n+1) to D(n) equals V(n) we set $V \equiv V(n)$ for every n. Now, $H(n) = H(n-1) \vee \{V^n d_1\} = D(n+1)$ and $R(n-1) \subset R(n) \cap D(n) \subset R(n-1) \vee \{V^n d_1\} \vee \{d_2\} = H(n)$, so it follows that

$$(12) \qquad \#(F_n) > 1 \Leftrightarrow R(n) \neq D(n) \Leftrightarrow R(n) \neq R(n) \cap D(n) \Leftrightarrow$$
$$\Leftrightarrow D(n) \neq R(n) \cap D(n) \Rightarrow R(n) \cap D(n) = R(n-1),\ n > 0.$$

These remarks lead to the following reformulation of the unicity condition.

We set $d_1(0) = d_1$, $d_2(0) = d_2$ and note that, if $n > 0$ and $\#(F_{n-1}) > 1$, the vectors $d_1(n), d_2(n)$ are well defined by the conditions

$$(13) \qquad d_1(n) \in R(n) \cap R(n-1)^{\perp},\ \|d_1(n)\| = 1,\ \langle V^n d_1, d_1(n)\rangle > 0,$$
$$d_2(n) \in D(n) \cap R(n-1)^{\perp},\ \|d_2(n)\| = 1,\ \langle d_2, d_2(n)\rangle > 0.$$

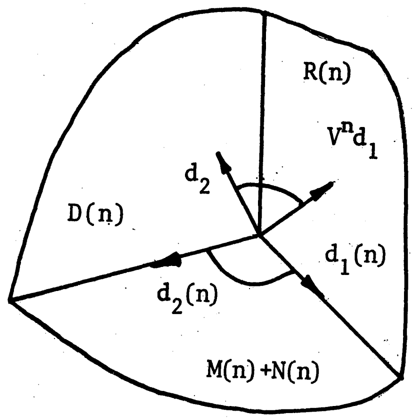
In such conditions, $R(n) \neq D(n)$ iff $d_1(n)$ and $d_2(n)$ are not colinear, thus motivating the following

DEFINITION. Set $\tilde{\gamma}_0 = \langle d_1, d_2\rangle$, and, if $n > 0$ and $\#(F_{n-1}) > 1$,
$$\tilde{\gamma}_n = \langle d_1(n), d_2(n)\rangle.$$

Then: $\#(F_{n-1}) > 1$ and $|\tilde{\gamma}_n| < 1 \Leftrightarrow \#(F_n) > 1$. Consequently,

$$(14) \qquad |\tilde{\gamma}_0| < 1, \dots, |\tilde{\gamma}_n| < 1 \Leftrightarrow \#(F_n) > 1;$$
$$|\tilde{\gamma}_0| < 1, \dots, |\tilde{\gamma}_{n-1}| < 1,\ |\tilde{\gamma}_n| = 1 \Leftrightarrow \#(F_{n-1}) > 1,\ \#(F_n) = 1.$$

The situation is as follows:



Remark that $|\tilde{\gamma}_n|$ measures the angle between the defect subspaces N(n) and M(n).

Moreover, $\rho(n) = \text{dist}^2[d_2, R(n)] = \text{dist}^2[\langle d_2, d_2(n)\rangle d_2(n), R(n)] =$
$= \langle d_2, d_2(n)\rangle^2 \text{dist}^2[d_2(n), R(n)] = \text{dist}^2[d_2, R(n-1)](1-|\tilde{\gamma}_n|^2)$, so:

(15) If $\#(F_{n-1}) > 1$ then $\rho(n) = \rho(n-1)(1-|\tilde{\gamma}_n|^2) = \Pi\{(1-|\tilde{\gamma}_j|^2): 0 \leqslant j \leqslant n\}$.

Now, (14) is precisely the fundamental property of the *Schur parameters* $\{\gamma_j\}$ associated to $f(z) = \Sigma\{c_j z^j: j \geqslant 0\}$, which are defined by the iteration formula

$$f_0 \equiv f \ , \ f_{j+1}(z) = \{f_j(z)-\gamma_j\}/\{z[1-\bar{\gamma}_j f_j(z)]\} \ , \ \gamma_j = f_j(0) \ , \ j \geqslant 0 \ ,$$

which is to be continued up to the first γ_n of modulus 1 ($\Rightarrow f_n(z) \equiv \gamma_n$), if any; each γ_n depends on $\hat{f}(k) = c_k$ for $k \leqslant n$. Thus, we are led to the conjecture

$$\tilde{\gamma}_n \equiv \gamma_n$$

More precisely, we shall prove that

THEOREM B. *Given a sequence* $\{c_n: n \geqslant 0\} \subset C$, *let* $\{\gamma_n: n \geqslant 0\}$ *be its Schur parameters and* $\{\tilde{\gamma}_n: n \geqslant 0\}$ *defined as above. Then:*

(i) $\{\gamma_n\}$ *is infinite iff* $\{\tilde{\gamma}_n\}$ *is infinite, i.e., iff* $\tilde{\gamma}_n < 1$

for every n, *and in such case* $\tilde{\gamma}_n \equiv \gamma_n$.

(ii) $|\gamma_0| < 1,\dots,|\gamma_n| < 1,\ |\gamma_{n+1}| = 1 \Longleftrightarrow |\tilde{\gamma}_0| < 1,\dots,$
$|\tilde{\gamma}_n| < 1,\ |\tilde{\gamma}_{n+1}| = 1 \Rightarrow \gamma_j = \tilde{\gamma}_j\ ,\ j = 0,\dots,n+1.$

Note that by means of the antilinear isometry J we can prove that $<V^n d_1, d_1(n)> = <d_2, d_2(n)> = \rho(n-1)^{1/2}$, $n \geq 1$.

Also: $\tilde{\gamma}_n = \rho(n-1)^{-1/2} <d_1(n), d_2>$, $n \geq 1$.

A FORMULA FOR $d_1(n)$, $n > 0$.

From $R(n) = V\{V^m d_1: m \leq 0\} \oplus V\{V^m d_2: m > n\} \oplus V\{V^p d_1, V^q d_2: 0 < p,q \leq n\}$ it follows that $d_1(n) \in R(n) \cap R(n-1)^{\perp}$ can be written as $d_1(n) = \Sigma\{\alpha_p(n) V^p d_1: 1 \leq p \leq n\} + \Sigma\{\beta_q(n) V^q d_2: 1 \leq q \leq n\}$ and is orthogonal to $V^p d_1$, $0 < p < n$, and to $V^q d_2$, $0 < q \leq n$. Thus:

(i) $\alpha_p(n) + \Sigma\{\beta_q(n)\bar{c}_{p-q}: 1 \leq q \leq p\} = 0\ ,\ 0 < p < n$,

(ii) $\Sigma\{\alpha_p(n) c_{p-q}: q \leq p \leq n\} + \beta_q(n) = 0\ ,\ 0 < q \leq n$.

Remembering that $<V^n d_1, d_1(n)> = \rho(n-1)^{1/2}$ we arrive at

(iii) $\alpha_n(n) + \Sigma\{\beta_q(n)\bar{c}_{n-q}: 1 \leq q \leq n\} = \rho(n-1)^{1/2}$.

Setting $\alpha(n) = \Sigma\{\alpha_j(n) e_{j-1}: 1 \leq j \leq n\}$,
$\beta(n) = \Sigma\{\beta_j(n) e_{j-1}: 1 \leq j \leq n\}$,
from (i) and (iii) we get $\alpha(n) = -\Gamma_{n-1}^{*}\beta(n) + \rho(n-1)^{1/2} e_{n-1}$, and, from (ii), $\beta(n) = -\Gamma_{n-1}\alpha(n)$. Consequently:

(16) $$d_1(n) = \Sigma\{\alpha_p(n) V^p d_1: 1 \leq p \leq n\} + \Sigma\{\beta_q(n) V^q d_2: 1 \leq q \leq n\}\ ,$$
with $$\alpha(n) = (\alpha_1(n)\dots\alpha_n(n)) = (1 - \Gamma_{n-1}^{*}\Gamma_{n-1})^{-1}\rho(n-1)^{1/2} e_{n-1}\ ,$$
$$\beta(n) = (\beta_1(n)\dots\beta_n(n)) = -\Gamma_{n-1}\alpha(n).$$

FORMULAS FOR $\tilde{\gamma}_n$.

The above shows that $\tilde{\gamma}_n = \rho(n-1)^{-1/2} <d_1(n), d_2> =$

$= \rho(n-1)^{-1/2}\Sigma\{\alpha_p(n)c_p: 1 \leqslant p \leqslant n\}$, so:

(17) $\tilde{\gamma}_n = <(I - \Gamma_{n-1}{}^*\Gamma_{n-1})^{-1}e_{n-1}, \Sigma\{\bar{c}_p e_{p-1}: 1 \leqslant p \leqslant n\}>$ if $n>0$, $\tilde{\gamma}_0 = c_0$.

Setting $\theta(n) = \theta_1 e_0 + \ldots + \theta_n e_{n-1} = (I - \Gamma_{n-1}{}^*\Gamma_{n-1})^{-1}e_{n-1}$, (16) shows that $\tilde{\gamma}_n = \theta_1 c_1 + \ldots + \theta_n c_n$, so Cramer's rule gives $\tilde{\gamma}_n$ as a quotient of determinants:

(18) $$\tilde{\gamma}_n = \tilde{D}_n/\tilde{\Delta}_n \quad \text{if} \quad n > 0 \quad , \quad \tilde{\gamma}_0 = c_0 ,$$

where $\tilde{\Delta}_n = \det(I - \Gamma_{n-1}{}^*\Gamma_{n-1})$ and $\tilde{D}_n$ is the determinant of the matrix obtained from the one of $(I - \Gamma_{n-1}{}^*\Gamma_{n-1})$ by replacing the last row by $c_1 \ldots c_n$ (so in particular $\tilde{D}_1 = c_1$).

We now show that we may assume that $c_0 \neq 0$. If $c_0 = \ldots = c_{t-1} = 0$ we set $c'_0 = c_t, \ldots, c'_k = c_{k+t}, \ldots$. Then, with obvious notation, the correspondence from $H(t+n)$ to $H'(n)$ given by

$$V^{t+p}d_1 \to V'^p d_1 , \ \forall\, p \leqslant n, \text{ and } V^q d_2 \to V'^q d_2 , \ \forall\, q \geqslant 0 ,$$

defines a unitary operator by means of which we can prove that $\tilde{\gamma}_{t+n} = \tilde{\gamma}'_n$, $\forall\, n \geqslant 0$. That is, $\tilde{\gamma}_{t+n}[z^t f(z)] = \tilde{\gamma}_n[f(z)]$. From Schur's original work we know that the same holds for the γ_n. So, in order to prove that $\tilde{\gamma}_n = \gamma_n$, we may assume that $c_0 \neq 0$.

APLICATION OF A FORMULA OF SCHUR.

In [Sch.] it is proved that $\gamma_n = -d_n/\delta_n$, with $\delta_n = \tilde{\Delta}_n$ and d_n the determinant of the matrix $M = (M_{jk})_{j,k=1,2}$, each $M_{jk} = [m_{jk}(r,s)]$ being an n by n matrix given as follows (with the non specified entries equal to zero):

$m_{11}(r,r-1) = 1$ for $2 \leqslant r \leqslant n$, $m_{11}(1,n) = c_0$;

$m_{12}(r,r+t-1) = c_t$ for $1 \leqslant r \leqslant n$ and $1 \leqslant t \leqslant n-r+1$, $m_{11}(r,r-1) = c_0$ for $2 \leqslant r \leqslant n$;

$m_{21}(r,r-t-1) = \bar{c}_t$ for $2 \leq r \leq n$ and $0 \leq t \leq r-2$, $m_{21}(1,n) = 1$; $m_{22}(r,r-1) = 1$ for $2 \leq r \leq n$.

Thus, $\det M_{11} = (-1)^{n-1}c_0$, and, if $c_0 \neq 0$, there exists M_{11}^{-1} and $M = PQ$, with $P = (P_{jk})_{j,k=1,2}$, $P_{11} = M_{11}$, $P_{12} = P_{21} = 0$, $P_{22} = I$ and $Q = (Q_{jk})_{j,k=1,2}$, $Q_{11} = I$, $Q_{12} = M_{11}^{-1} M_{12}$, $Q_{21} = M_{21}$, $Q_{22} = M_{22}$. From a lemma also due to Schur it follows that $\det Q = \det(M_{22}-M_{21}M_{11}^{-1}M_{21})$ and consequently

$$d_n = (-1)^n c_0 \det(M_{22}-M_{21}M_{11}^{-1}M_{12}).$$

Now, $M_{11}^{-1}M_{12} = [x(r,s): 1 \leq r,s \leq n]$ is such that $[x(r,s): 1 \leq r,s \leq n-1]$ is the matrix of Γ_{n-2} and $x(n,s) = c_0^{-1}c_s$, $1 \leq s \leq n$, $x(r,n) = c_{n-r}$, $1 \leq s \leq n-1$, while $[m_{12}(r+1,s): 1 \leq r,s \leq n-1]$ is the matrix of Γ_{n-2}^*, and $m_{12}(1,s) = m_{12}(s+1,n) = 0$ for $1 \leq s \leq n-1$, $m_{12}(1,n) = 1$. Thus, $M_{21}M_{11}^{-1}M_{12} = [y(r,s): 1 \leq r,s \leq n]$ has the following form: $[y(r+1,s): 1 \leq r,s \leq n-1]$ is the matrix of $\Gamma_{n-2}^*\Gamma_{n-2}$; $y(1,s) = c_0^{-1}c_s$, $1 \leq s \leq n$; $y(r+1,n) = \Sigma\{\bar{c}_{r-1-j}c_{n-1-j}: 0 \leq j \leq r-1\}$, $1 \leq r \leq n-1$.

Now, remembering the definition of $\tilde{D}_n$, it is not difficult to see that $d_n = (-1)^n c_0 \det(M_{22}-M_{21}M_{11}^{-1}M_{12}) = \tilde{D}_n$. Thus, theorem B has been proved.

In a sequel to this paper, our approach will be related with entropy considerations.

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Recibido en febrero de 1989.