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SPATIAL TEMPORAL RESPONSE FUNCTIONS OF PLASMA ELECTRODYNAMICS

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The polarization response tensor, for a uniform plasma without fields which may contain beams, is obtained in the space-time representation considering the charges induced by delta-like electric fields. The derivation and a discussion of the properties, for several one and three dimensional cases, does not rely on Fourier transforms and is given directly in the $\bar{x}$, t description. The integro-differential equations of plasma electrodynamics are set up with these responses. A formalization of the method and its connection with other electrodynamical concepts are also presented.

1. INTRODUCTION

Physical systems with the properties of causality, linearity, space homogeneity and time invariance are characterized by integral constitutive equations of the convolution type ([1]). Response and excitation in these systems are linked by general expressions of the form

$$\bar{P}(\bar{x},t) = \int_{-\infty}^{t} d\tau \int d^3\xi \; h(\bar{x}-\bar{\xi},t-\tau).\bar{E}(\bar{\xi},\tau) \; . \qquad (1)$$

Here we have written the polarization $\bar{P}(\bar{x},t)$ as a functional

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of the electric field $\bar{E}(\bar{x},t)$ in a plasma([2]).The kernel $h(\bar{x},t)$ is called the polarization response tensor. Due to causality, $h(\bar{x},t) = 0$ for $t < 0$. When $\bar{E}(\bar{x},t)$ is known, it gives the inner plasma currents through the equation $\partial\bar{P}(\bar{x},t)/\partial t = \bar{j}(\bar{x},t)$. The $\bar{k},\omega$ Fourier transform $F_{\bar{k},\omega}[h(\bar{x},t)] = \chi(\bar{k},\omega)$, is the complex susceptibility tensor, a fundamental quantity in plasma electrodynamics, related to the dielectric tensor by $\epsilon(\bar{k},\omega) = I + 4\pi\chi(\bar{k},\omega)$. For colisionless plasmas, $\chi(\bar{k},\omega)$ is traditionally derived by perturbative calculations from Vlasov and Maxwell equations, working in the $\bar{k},\omega$ space. In principle $h(\bar{x},t)$ can be calculated from $\chi(\bar{k},\omega)$ performing inverse Fourier transforms. However, in few problems the real $\bar{x}$, t dynamical behaviour is brought forth due to its natural complexity.

We have noted recently that the one dimensional, longitudinal response function h(x,t) can be derived directly by simple physical arguments([3]).This approach may be called the percussion method and can be extended to other cases. In mathematical terms it consists, essentially, in the determination of an $\bar{x},t$ Green's function for the inverse polarization operator. The procedure does not pass through the $\bar{k},\omega$ space. In some cases the Vlasov equation is not needed to derive the polarization response by this method, although basically the same physical information is used. In the case of a non-homogeneous plasma, formed by radially focused beams, the r,t polarization response function was similarly obtained([4]).We found it useful for setting up the integral equation for the electrostatic modes of the system. The percussion approach shows in a clear way the basic physical elements that produce the plasma polarization.

In this paper we study the spatial-temporal polarization response tensor of a uniform plasma, without externally imposed fields. We give the three dimensional response $h(\bar{x},t)$ for isotropic distribution functions $f_o(v^2)$. For non isotropic plasmas we derive, for simplicity, the averaged longitudinal $H_{yy}(y,t)$ and transverse $H_{zz}(y,t)$ components for excitations

$E_{y,z}(y,t)$ with planar symmetry. The responses H_{yy} and H_{zz}, functions of one space variable, y, are the integrals of h_{yy} and h_{zz}, respectively, over the other two variables x, z. These results have been derived taking into consideration the possible presence of beams in the plasma. The paper is organized as follows. In section 2 we describe the method and its relation with the main concepts of plasma electrodynamics. In section 3 and 4 we obtain the one dimensional responses H_{yy}, H_{zz} and discuss their properties. In section 5 we prove the equivalence of the k,ω transform of H_{zz} with the transverse susceptibility $\chi^t(k,\omega)$. Section 6 is devoted to the three dimensional response tensor for isotropic plasmas. We present an explicit form of the integro-differential equation for the electric field in this case. In section 7 we show how to formalize the percussion approach with the integration of the Vlasov equation along trajectories.

2. IMPULSIVE ELECTRIC FIELDS

In the $\bar{k},\omega$ representation, Maxwell equations are written as ([5])

$$\Lambda(\bar{k},\omega).\bar{E}(\bar{k},\omega) = -(4\pi i/\omega)\ \bar{j}_e(\bar{k},\omega)\ , \qquad (2)$$

where Λ is the Maxwell operator defined by $\Lambda = \Lambda^\circ + 4\pi\chi$, $\Lambda^\circ = I + (c^2/\omega^2)\ (\bar{k}\bar{k} - k^2 I)$ is the Maxwell vacuum operator and $\bar{j}_e$ are the external currents. If we define a vacuum tensor $L^\circ(\bar{x},t) = F^{-1}_{\bar{x},t}\,[\Lambda^\circ(\bar{k},\omega)]$, eq.(2) can be written in the $\bar{x},t$ representation as

$$\int_{-\infty}^{t} d\tau \int d^3\xi\ L^\circ(\bar{x}-\bar{\xi},t-\tau).\bar{E}(\bar{\xi},\tau) = -4\pi\,[\bar{q}_e(\bar{x},t) + \bar{P}(\bar{x},t)]\ , \qquad (3)$$

where $\bar{q}_e(\bar{x},t) = \int_{-\infty}^{t} \bar{j}_e(\bar{x},t')dt'$.

For electrostatic perturbations with rot $\bar{E} \approx 0$, eq.(3) reduces to

$$\bar{E}(\bar{x},t) = -4\pi\,[\bar{q}_e(\bar{x},t) + \bar{P}(\bar{x},t)] \ . \qquad (4)$$

Thus any electrostatic field can be produced by a proper combination of external and inner polarization charges. In particular, the percussion method requires localized and impulsive fields of the form $E^{\circ}_j(\bar{x},t) = \mathcal{E}^{(j)}\delta(\bar{x})\ \delta(t)\bar{e}_j$, where $\mathcal{E}^{(j)}$ is the electric field strength and $\bar{e}_j$ is the unit vector in the x_j direction, j=1,2,3. Using these fields in eq.(1) we get

$$P_i^{(j)}(\bar{x},t) = h_{ij}(\bar{x},t)\ \mathcal{E}^{(j)} \ . \qquad (5)$$

Thus, by computing the polarizations $P_i^{(j)}$ produced by delta-like electric fields E°_j, we obtain the components of the polarization response tensor. The impulsive field gives a percussion to the plasma particles which modifies their velocity in $\Delta\bar{v}$, without changing their position. The corresponding alteration of the distribution function is easy to evaluate. After the percussion the particles flow freely, in the absence of any perturbative field, hence the current and the polarization can be computed.

It must be noted (from eq.(4)) that the solitary pulse of the electric field in the plasma can be achieved only by the artificial inclusion of a set of external charges, so that $\bar{q}_e(\bar{x},t) = -\bar{P}^{(j)}(\bar{x},t)$, for all $t > 0$. When $h(\bar{x},t)$ is known, eq.(4) together with eq.(1) can be used to find the system of external charges necessary to produce any particular electrical field in the plasma, according to the equation

$$-4\pi\bar{q}_e = \bar{E}(\bar{x},t) + 4\pi\int_{-\infty}^{t} d\tau \int d^3\xi\ h(\bar{x}-\bar{\xi},t-\tau).\bar{E}(\bar{\xi},\tau) \ . \qquad (6)$$

Conversely, eq.(6) can be regarded as the integral equation for the determination of the electric field $\bar{E}(\bar{x},t)$ in electrostatic problems, with or without external charges.

The same ideas can be extended to the treatment of electromagnetic perturbations, where eq.(3) takes now the place of eq. (6). The calculation of the polarizations, eq.(5), is more involved, because an induced magnetic field accompanies the im-

pulsive electric field. This magnetic field produces additional polarization currents, as we show in section 4.

Summarizing, the response tensor $h(\bar{x},t)$ is an auxiliary concept which can be used: (a) to find the $\bar{x},t$ distribution of inner charges induced in the plasma in any electrodynamical problem when the electric field is given; (b) to set up the integro-differential equation for $\bar{E}(\bar{x},t)$ in initial and boundary problems with assigned external charges. The peculiar feature of the percussion method is that a Fourier representation is not needed for the derivation of $h(\bar{x},t)$.

3. PERCUSSION AND LONGITUDINAL RESPONSE

We analyze the polarization produced by two electric fields $E^{\circ}_{y,z} = \mathcal{E}\,\delta(y)\,\delta(t)$, in a uniform plasma, without external magnetic field. These planar excitations are related to one dimensional responses, according to eq.(1), $P^{\circ}_{y}(y,t) = \mathcal{E}\,H_{yy}(y,t)$, $P^{\circ}_{z}(y,t) = \mathcal{E}\,H_{zz}(y,t)$, where $H_{ij}(y,t) = \int\!\!\int dx\,dz\; h_{ij}(\bar{x},t)$.

Let us take first the pulse E°_{y}, normal to the excitation plane. This is an electrostatic case due to the symmetry. We introduce the set of rectangular functions δ_{ε}, such that $\delta_{\varepsilon} = 1/2\,\varepsilon$ for $|y| \leqslant \varepsilon$ and $\delta_{\varepsilon} = 0$ when $|y| > \varepsilon$, as approximants of $\delta(y)$. For brevity we consider the perturbation of one species of particles only. The electrons within the layer $|y| \leqslant \varepsilon$ at $t=0$ suffer a jump $\Delta v_y = -(e/m)\mathcal{E}/2\varepsilon$ in the velocity component v_y. This increment produces a charge per unit area $\tilde{\sigma}(\bar{v}) = en_{\circ}[f_{\circ}(v_x,v_y,v_z) - f_{\circ}(v_x,v_y+\Delta v_y,v_z)]2\varepsilon$ originated from particles with velocity $\bar{v}$, per unit of velocity space, situated in $|y| \leqslant \varepsilon$ ($f_{\circ}(\bar{v})$ is the distribution function of the electrons normalized to unity, $n_{\circ}$ in the electron density). The first term represents the uncompensated charge of the background ions, left by the electrons with initial velocity v_y. The second term is due to new electrons, which after the percussion assume the velocity v_y. If the perturbation is small

we have $\tilde{\sigma} = -\&(\omega_p^2/4\pi)\partial f_o/\partial v_y$ to first order in the pulse intensity $\&$ (ω_p is the plasma frequency). These particles move then with different constant velocities and constitute a set of planes travelling charged layers. Neglecting the layer width ε, we write for $t > 0$

$$\sigma(\bar{v},t) = -\&(\omega_p^2/4\pi)\ \delta(y - v_y t)\ \partial f_o/\partial v_y\ , \tag{7}$$

for the charge density, per unit of velocity space, moving with velocity v_y. The current density j_y at y,t is therefore

$$j_y(y,t) = -\&(\omega_p^2/4\pi)\ (y/t^2)\ \partial F_o/\partial v_y|_{v_y=y/t}\ , \text{ for } y \neq 0\ ,$$

$$= \&(\omega_p^2/4\pi)\ \delta(y)\ A\ , \text{ for } y = 0\ ,$$

where $F_o(v_y) \equiv \int dv_x\ dv_z\ f_o(\bar{v})$ and $A \equiv \lim_{\varepsilon\to 0} \int_{|v_y|<\varepsilon} dv_y\ F_o(v_y)$.

The factor A is the fraction of electrons with $v_y = 0$ (this includes, for instance, the case of beams along x,z with $v_y = 0$). The time integral of the current at a fixed position y gives the polarization

$$P_y^o(y,t) = \&(\omega_p^2/4\pi)\ F_o(y/t)\ , \text{ for } y \neq 0\ ,$$

$$= \&(\omega_p^2/4\pi)\ A\ \delta(y)\ t\ , \text{ for } y = 0.$$

Comparing with $\& H_{yy}$ we finally get

$$H_{yy}(y,t) = (\omega_p^2/4\pi)\ F_o(y,t)\ ,\ \text{for } y \neq 0\ ,$$

$$= (\omega_p^2/4\pi)\ A\ t\ \delta(y)\ ,\ \text{for } y = 0. \tag{8}$$

For a cold beam with velocity $\bar{V}^o$, $f_o(\bar{v}) = (\bar{v} - \bar{V}^o)$. Then,

$$H_{yy}(y,t) = (\omega_p^2/4\pi)\ \delta(y - V_y^o t)\ t. \tag{9}$$

In Ref. [3] we have shown the equivalence of eq. (8) with the formula that can be derived starting from the well-known longitudinal susceptibility $\chi^{\ell}(k,\omega)$. Therefore we do not elabo-

rate further on this result, except for a comment on the configuration of charges for this case. The plasma charges and currents are given by $\rho = -\& \, \partial H_{yy}/\partial y$, $j = \& \, \partial H_{yy}/\partial t$, respectively. However, in order to produce the impulsive field, the following external charges and currents are also needed

$$\rho_e = (\&/4\pi)[\delta'(y)\delta(t) + 4\pi \, \partial H_{yy}/\partial y] \ ,$$

$$j_e = -(\&/4\pi)[\delta(y)\delta'(t) + 4\pi \, \partial H_{yy}/\partial t] \ .$$

For a symmetric function $F_o(v_y) = F_o(-v_y)$ (e.g. a Maxwellian) there are neither polarization currents nor charges at y=0, for all t. Thus, to have a delta-like electric field we need only a dipolar layer of external charges at y=0, pulsed in time. We may conceive a layer of neutral atoms which are polarized during a short time and return to neutrality. External charges must then be added at all times, but this is of no consequence for the plasma since we assume that the interaction of charged particles is only through the self-consistent field.

4. THE RESPONSE TO ELECTROMAGNETIC EXCITATION

We consider now a field E_z^o, contained in the percussion plane. Associated to it there is a magnetic field $B_x^o = -c\&\delta'(y)$ for $t \geqslant 0$. The particles' acceleration is, for $t \geqslant 0$,

$$\bar{a}(y,t) = -(e/m)\&[\delta(t)\delta(y)\bar{e}_z + \delta'(y)(v_y\bar{e}_z - v_z\bar{e}_y)] \ . \qquad (10)$$

The particles that pass through y=0 experiment changes of velocity at all times. We introduce here also the set of functions δ'_ε , such that $\delta'_\varepsilon = 1/\varepsilon^2$ for $-\varepsilon \leqslant y < 0$, $\delta'_\varepsilon = 0$ at y=0, $\delta'_\varepsilon = -1/\varepsilon^2$ for $0 < y \leqslant \varepsilon$ and $\delta'_\varepsilon = 0$ when $|y| > \varepsilon$, as approximants for $\delta'(y)$.

The jump Δv_z is formed by two terms: $\Delta v_{z1} = -(e/m)\&/2\varepsilon$ of electric origin as before, for those particles within $|y| \leqslant \varepsilon$ at t=0, and Δv_{z2} due to the magnetic terms. Let us take first

$v_y > 0$ and $t > 0$. When an electron crosses the layer $(-\varepsilon,0)$ it suffers a velocity increment $\Delta v_{z2} = -(e/m)\&\int dt'(v_y/\varepsilon^2)$, where the limits of integration are t and $t + \varepsilon/v_y$. Thus, $\Delta v_{z2} = -(e/m)\&/\varepsilon$. After crossing the following layer $(0,\varepsilon)$ it loses the same amount Δv_{z2}. The same can be said for electrons with $v_y < 0$, for $t > 0$. We conclude that particles with $v_y \neq 0$ gain Δv_{z2} at $y = 0$ and lose instantly the same increment when they leave the position $y = 0$. However, there also exist particles with $v_y \neq 0$ which are located at $y = 0$ for $t = 0$. These suffer the influence of half δ'_ε only, so that for them $\Delta v_{z2} = (e/m)\&/\varepsilon$. These particles do not lose their velocity increment.

For the variation of the v_y component we obtain

$$v_y = -(e/m)\&\left[\int_t^{t+\varepsilon/v_y} dt'\ v_z/\varepsilon^2 + \int_{t+\varepsilon/v_y}^{t+(\varepsilon/v_y)+(\varepsilon/v'_y)} dt'\ v'_z(-1/\varepsilon^2)\right] =$$

$$= -(e/m)\&[(v_z/v_y) - (v'_z/v'_y)]/\varepsilon \quad ,$$

for $v_y \neq 0$, $t > 0$. Since Δv_z, Δv_y depend linearly on &, therefore within the linear approximation $v'_z/v'_y = v_z/v_y$. Hence we can say that the electrons gain $\Delta v_y = -(e/m)\&v_z/v_y\varepsilon$ when they cross $y=0$, and lose it immediately. Considering as before particles with $v_y \neq 0$, located at $y=0$ for $t=0$, we find that they gain $\Delta v_y = (e/m)\&v_z/v_y\varepsilon$ from half of δ'_ε and this increment is not lost thereafter.

We have seen in the previous section that velocity jumps at $y=0$ generate charged layers. The layers created at time t have a charge per unit area

$$\tilde{\sigma}(\bar{v},t) = -en_o 2\varepsilon\Delta v_{z1}\ \partial f_o/\partial v_z \quad , \quad \text{for } t = 0 \quad ,$$

$$= -en_o\varepsilon[\Delta v_{z2}\ \partial f_o/\partial v_z + \Delta v_y\ \partial f_o/\partial v_y] \quad , \quad \text{for } t > 0 \ .$$

When we account for the motion of these layers, the charge density per unit of velocity space (neglecting the width ε) is given by

$$\sigma(y,\bar{v},t) = -(\omega_p^2/4\pi)\{\delta(y)\ \partial f_o/\partial v_z + [\delta(y-v_y t) - \delta(y)](v_z/v_y)\ \partial f_o/\partial v_y\}, \quad (11)$$

when $v_y \neq 0$. In deriving this equation note that the contributions of Δv_{z1} and Δv_{z2} cancel each other when $y \neq 0$. Two terms contribute only at $y = 0$ and they correspond to the charge and discharge of layers due to velocity jumps of the particles that cross the δ'_ε influence region, as commented above. This is a process that takes place continuously in time.

To complete the analysis, consider now electrons with $v_y = 0$ at $t = 0$. For these we get $\Delta v_{z1} = -(e/m)\mathcal{E}/\varepsilon$ and $\Delta v_{z2} = 0$ to first order in $\mathcal{E}$, while $\Delta v_y = a_y t = (e/m)\mathcal{E} v_z t \delta'(y)$. Thus the contribution to the charge density of the particles with $v_y = 0$ is

$$\sigma(y,\bar{v},t) = -\mathcal{E}(\omega_p^2/4\pi)[\delta(y)\ \partial f_o/\partial v_z - \delta'(y)\ v_z t\ \partial f_o/\partial v_y] \quad (12)$$

Now we compute j_z from eqs.(11) and (12). We have

$$\begin{aligned} j_z(y,t) &= \int d^3v\ \sigma(y,\bar{v},t) v_z = \mathcal{E}(\omega_p^2/4\pi)\{\delta(y) P\int dv_y [F_o(v_y) + g'(v_y)/v_y] - \\ &- P\int dv_y \delta(y-v_y t) g'(v_y)/v_y + \delta(y) \lim_{\varepsilon\to 0} \int_{|v_y|<\varepsilon} dv_y\ F_o(v_y) + \\ &+ \delta'(y) t \lim_{\varepsilon\to 0} \int_{|v_y|<\varepsilon} dv_y\ g'(v_y)\} = \\ &= \mathcal{E}(\omega_p^2/4\pi)\{\delta(y)[1 + P\int dv_y\ g'(v_y)/v_y] + \delta'(y)\ t\ B\}\ , \quad y = 0\ , \\ &= -\mathcal{E}(\omega_p^2/4\pi)\ g'(y/t)/y\ , \quad y \neq 0. \end{aligned} \quad (13)$$

We introduced here the function $g(v_y) \equiv \int dv_z dv_x v_z^2 f_o(\bar{v})$ and the constant $B \equiv \lim_{\varepsilon\to 0} \int_{|v_y|<\varepsilon} dv_y\ g'(v_y) = g(0^+) - g(0^-)$. The symbol P denotes principal value of the integral. We observe that this integral is finite under the assumption $g'(0^+) = g'(0^-)$. We shall consider this valid in the rest of the paper. With another integration over time we obtain P_z^o. The final result is

$$H_{zz}(y,t) = (\omega_p^2/4\pi)\,[\delta(y)t(1 + P\int dv_y\, g'(v_y)/v_y) + \tfrac{1}{2}\,\delta'(y)t^2 B] \quad , \quad y = 0 \, ,$$

$$= (\omega_p^2/4\pi) \int_{-\infty}^{y/t} dv_y\, g'(v_y)/v_y^2 \quad , \quad y < 0 \quad , \qquad (14)$$

$$= -(\omega_p^2/4\pi) \int_{y/t}^{\infty} dv_y\, g'(v_y)/v_y^2 \quad , \quad y > 0 \, .$$

It is clear from eqs.(14) that the response is zero when $|y/t|$ is larger than the maximum velocity of the particles in the plasma. As $t \to \infty$, for $y \neq 0$, H_{zz} tends to a constant value since there are no charged layers left to pass through the plane y = constant.

We may note that if the plasma is cold in the variable v_z, i.e. if f_o has a $\delta(v_z)$ factor, then g = 0 and H_{zz} is zero except at the origin, where it is given by $(\omega_p^2/4\pi)\delta(y)t$. The k,ω Fourier transform of this expression leads to $4\pi\chi(\omega) = -\omega_p^2/\omega^2$, the familiar result of cold plasmas. From eqs.(11) and (12) it is easy to see that $H_{yz} = H_{xz} = 0$ when $\int dv_z v_z f_o = 0$. If, in addition, we assume that $\int dv_x v_x f_o = 0$, i.e. in the absence of steady state currents parallel to the plane of excitation, we find $H_{yz} = H_{xz} = H_{xy} = 0$. Thus, the one dimensional response tensor is diagonal with H_{yy} given by eq.(8), H_{zz} by eqs.(14) and H_{xx} has expressions similar to H_{zz}, with a function $k(v_y) = \int dv_x\, dv_z\, v_x^2 f_o(\bar{v})$ in place of $g(v_y)$.

In the presence of a cold beam, $f_o(\bar{v}) = \delta(\bar{v} - \bar{V}^o)$, we obtain (taking $V_y^o > 0$)

$$H_{zz}(y,t) = (\omega_p^2/4\pi)(V_z^{o2}/V_y^{o3})\theta(y)\,[V_y^o\delta(V_y^o - y/t) - 2\theta(V_y^o - y/t)], \quad y \neq 0 \, , \qquad (15)$$

$$= (\omega_p^2/4\pi)\,[1 + V_z^{o2}/V_y^{o2}]\;\delta(y)\;t \quad , \quad y = 0 \, .$$

Here $\theta(y)$ is the Heaviside (or step) function.

For an isotropic plasma, $f_o = f_o(v^2)$, we have

$P\int dv_y\ g'(v_y)/v_y = -1 + A$. Therefore,

$$H_{zz}(y=0,t) = (\omega_p^2/4\pi)\ \delta(y)\ t\ A \qquad (16a)$$

In the absence of a cold component with $v = 0$, $H_{zz}(y=0,t) = 0$. When $y \neq 0$, instead, we find

$$H_{zz}(y,t) = (\omega_p^2/4\pi) \int_{|y|<t}^{\infty} dv_y\ F_o(v_y^2)/v_y\ . \qquad (16b)$$

We conclude with a remark of a mathematical nature. Equations (8) and (14) hold for distribution functions which are linear combinations of functions of the type $f_o(\bar{v}) = \phi_1(\bar{v}) +$ $+ \phi_2(\bar{v})\theta(v_x-V_x)\theta(v_y-V_y)\theta(v_z-V_z) + \phi_3(\bar{v})\delta(v_y-U_y) + \phi_4(\bar{v})\ \delta(v_y-U_y)$ $\times\ \delta(v_z-U_z) + \phi_5\delta(\bar{v}-\bar{U})$, and similar ones. The functions ϕ_1, ϕ_2, ϕ_3, ϕ_4 are continuous and ϕ_5 is a constant. The second term describes jumps in one or more velocity components and the third, fourth and fifth ones allow for the presence of concentrated constituents or beams in one, two or three velocity components.

5. THE TRANSVERSE SUSCEPTIBILITY

We obtain here the complex transverse susceptibility $\chi^t(k,\omega)$ by transforming Fourier the response function H_{zz}. Note that $H(y,t) = F^{-1}_{y,t}[\chi(0,k,0)]$. If we set

$$G_-(v) \equiv \int_{-\infty}^{v} du\ g'(u)/u^2 \quad \text{and} \quad G_+(v) \equiv \int_{v}^{\infty} du\ g'(u)/u^2\ ,$$

we may write

$$4\pi\ \chi(k,\omega) = -(\omega_p^2/\omega^2)[1+P\int dv g'(v)/v] + (\omega_p^2/\omega^3)kB + \omega_p^2\int_{-\infty}^{\infty} dt\ e^{i\omega t} \times$$
$$\times\ \theta(t) \int_{-\infty}^{0^-} dy\ G_-(y/t)e^{-iky} - \int_{0^+}^{\infty} dy\ G_+(y/t)e^{-iky}\ ].$$

On the other hand we have

$$\int_{-\infty}^{\infty} dt\ e^{i\omega t}\ \theta(t) \int_{0^+}^{\infty} dy\ e^{-iky}\ G_+(y/t) =$$

$$= \int_{0^+}^{\infty} dv\ v\ G_+'(v)/\omega(\omega-kv) + [v\ G_+(v)/\omega^2]_{v=0^+} =$$

$$= -[\int_{0^+}^{\infty} dv g'(v)/v + k\int_{0^+}^{\infty} dv\ g'(v)/(\omega-kv) - [v\ G_+(v)]_{v=0^+}]/\omega^2$$

and similarly

$$\int_{-\infty}^{\infty} dt\ e^{i\omega t}\ \theta(t) \int_{-\infty}^{0^-} dy\ e^{-iky}\ G_-(y/t) =$$

$$= [\int_{-\infty}^{0^-} dv\ g'(v)/v + k\int_{-\infty}^{0^-} dv\ g'(v)/(\omega-kv) - [v\ G_-(v)]_{v=0^-}]/\omega^2 .$$

Since $[vG_+]_{v=0^+} + [vG_-]_{v=0^-} = 0$, when $g'(0^+) = g'(0^-)$ (which is the condition for the finiteness of $P\int dv\ g'(v)/v$) and noting that $B = \lim_{\varepsilon\to 0} \omega \int_{|v|<\varepsilon} dv\ g'(v)/(\omega-kv)$, we obtain

$$4\pi\ \chi^t(k,\omega) = -(\omega_p/\omega)^2 [1 - k \int_{-\infty}^{\infty} dv\ g'(v)/(\omega-kv)]$$

which coincides with the traditional result ([5]).

6. A THREE DIMENSIONAL RESPONSE

For isotropic plasmas, the polarization effect of the induced magnetic field is null and the calculation of h is straightforward, as in the electrostatic case of section 3. Let us take a point-like excitation $\bar{E} = \&\delta(t)\ \delta(\bar{x})\ \bar{e}_z$, with induced magnetic components $B_x = -c\&\theta(t)\delta(x)\delta'(y)\delta(z)$, $B_y = c\&\theta(t)\delta'(x)\delta(y)\delta(z)$, $B_z = 0$. The particles that pass through the origin for $t > 0$ experiment velocity increments

$$\Delta v_x = (e/m)\&\int dt\ v_z\delta'(x)\delta(y)\delta(z)\ \theta(t) ,$$

$$\Delta v_y = (e/m)\&\int dt\ v_z\delta(x)\delta'(y)\delta(z)\ \theta(t) ,$$

$$\Delta v_z = -(e/m)\int dt\,[v_x \delta'(x)\delta(y) + v_y \delta(x)\delta'(y)]\ \delta(z)\ \theta(t)\ .$$

The perturbation of the distribution function in the linear approximation is $\Delta f = \Delta v_x\ \partial f_o/\partial v_x + \Delta v_y\ \partial f_o/\partial v_y + \Delta v_z\ \partial f_o/\partial v_z$. However, when $f_o = f_o(v^2)$ we have $\partial f_o/\partial v_x = 2v_x\ \partial f_o/\partial v^2$ and similar formulas for y and z. Thus we immediately see that $\Delta f = 0$, when $t > 0$. For $t=0$ there is the effect of the electric field only and considerations similar to those of section 3 lead to a charge density ρ, per unit of velocity space, given by

$$\rho(\bar{x},\bar{v},t) = -(\omega_p^2/4\pi)\ \delta(\bar{x} - \bar{v}t)\ \partial f_o/\partial v_z\ . \qquad (17)$$

It is possible to show that the one dimensional responses can also be obtained from this formula. For instance, taking $h_{zz}(\bar{x},t)$ and integrating over x and z we get

$$H_{zz}(0,t) = -(\omega_p^2/4\pi)\ \delta(y) \int dx\ dz \int_0^t dt' \lim_{\varepsilon\to 0} \int_{|v_y|<\varepsilon} dv_y \quad x$$

$$x \int dv_x\ dv_z\ \delta(x-v_x t)\ \delta(z-v_z t)\ v_z\ \partial f_o/\partial v_z =$$

$$= (\omega_p^2/4\pi)\ \delta(y)\ t\ A\ ,$$

$$H_{zz}(y,t) = -(\omega_p^2/4\pi) \int dx\ dz \int_0^t dt' \int d^3v\ \delta(\bar{x}-\bar{v}t)\ v_z\ \partial f_o/\partial v_z =$$

$$= (\omega_p^2/4\pi) \int_0^t dt'\ F_o(y^2/t'^2)/t'\ ,\quad y \neq 0\ .$$

With a change of variable these expressions coincide with eqs. (16) derived in section 4 for an isotropic plasma. Starting from ρ for a delta-like electric field oriented along $\bar{e}_y$ and following a similar procedure we obtain the longitudinal response, eq.(8).

It is interesting to find the explicit form of h for three dimensional excitations. Returning to eq.(17) we have

$$h_{zz}(\bar{x},t) = -2(\omega_p^2/4\pi) \int_0^t \frac{dt'}{t'^3} \int d^3v\ \delta(r/t - v)\ v_z^2\ \partial f_o/\partial v^2 \quad =$$

$$= -(\omega_p^2/4\pi)(z^2/r^4) \int_{(r/t)^2}^{\infty} dv^2\ v^2\ \partial f_o/\partial v^2 \ , \text{ for } |\bar{x}|=r\neq 0 \ ,$$

$$h_{zz}(0,t) = (\omega_p^2/4\pi)\ \delta(\bar{x})\ t\ C \ ,$$

where C is the fraction of particles with zero velocity. Completing similar calculations we prove the result

$$h_{ij}(\bar{x},t) = -(\omega_p^2/4\pi)\ (x_i x_j/r^4) \int_{(r/t)^2}^{\infty} dv^2\ v^2\ \partial f_o/\partial v^2 \ , \text{ for } r \neq 0 \ ,$$

$$= (\omega_p^2/4\pi)\ \delta(\bar{x})\ t\ C\ \delta_{ij} \quad , \quad \text{for } r = 0 \ . \tag{18}$$

Here δ_{ij} is the Kronecker delta.

The integro-differential equation for the electric field in a plasma can be written as

$$\frac{\partial^2 \bar{E}}{\partial t^2} + c^2\ \text{rot rot}\ \bar{E} = -4\pi \int d^3\xi \int_0^{\infty} d\tau\ h(\bar{\xi},\tau).\frac{\partial^2 \bar{E}}{\partial \tau^2}(\bar{x}-\bar{\xi},\bar{t}-\tau) - 4\pi\ \frac{\partial \bar{j}_e}{\partial t} \ . \tag{19}$$

For an isotropic plasma and assuming that the electric field and its first derivative are zero for $t \to -\infty$ and that $\lim_{v\to\infty} v^5 f_o'(v^2) = 0$, we may rearrange the equation as

$$\frac{\partial^2 \bar{E}}{\partial t^2} + c^2\ \text{rot rot}\ \bar{E} = -\omega_p^2 \int d^3\xi\ (\bar{\xi}\ \bar{\xi}/\xi^6) \int_0^{\infty} d\tau\ W(\xi^2/\tau^2).\bar{E}(\bar{x}-\bar{\xi},t-\tau) -$$

$$- 4\pi\ \frac{\partial \bar{j}_e}{\partial t} - \omega_p^2\ C\ \bar{E}(\bar{x},t) \ , \tag{20}$$

where ξ stands for $|\bar{\xi}|$ and $W(v^2) = 10v^6 f_o'(v^2) + 4v^8 f_o''(v^2)$. For a Maxwellian distribution function $f_o^M = (a/\pi)^{3/2}\ \exp(-av^2)$ and the function W takes the particularly simple form $W^M(v^2) = (a/\pi)^{3/2}\exp(-av^2)\ (4v^8a^2 - 10\ v^6a)$ and C = 0. When the plasma is cold, the integral part of eq.(20) does not contribute and C=1. For a monoenergetic isotropic distribution we have $f_o(v^2) = \delta(v^2 - V_o^2)/2\pi V_o$. In this case,

$$h(\bar{x},t) = (\omega_p^2/8\pi^2 V_o)(\bar{x}\,\bar{x}/r^4)[\theta(V_o - r/t) + (r/t)^2\delta(r^2/t^2 - V_o^2)]$$

and, therefore,

$$\frac{\partial^2 \bar{E}}{\partial t^2} + c^2 \text{ rot rot } \bar{E} = -(\omega_p^2/2\pi V_o)\int d^3\xi(\bar{\xi}\bar{\xi}/\xi^4).\{\frac{\partial}{\partial t}\bar{E}(\bar{x}-\bar{\xi},t-\xi/V_o) + 2(\xi/V_o)\frac{\partial^2}{\partial t^2}\bar{E}(\bar{x}-\bar{\xi},t-\xi/V_o)\} - 4\pi\frac{\partial \bar{j}_e}{\partial t} \quad . \tag{21}$$

A sum of distributions of this type, with proper weights, may represent a discretization of an isotropic distribution function.

7. PERCUSSION AND INTEGRATION ALONG ORBITS

We formalize here the percussion procedure used in the derivation of the response functions. For definiteness we take the case of section 4, which includes the simpler cases of sections 3 and 6. The linearized Vlasov equation can be written as

$$\frac{d}{dt}f_1 = -\bar{a} \cdot \partial f_o/\partial \bar{v} \tag{22}$$

where f_1 is the perturbed distribution function and the time derivative is taken along the orbits

$$\bar{x} = \bar{x}_o + \bar{v}_o t \quad , \quad \bar{v} = \bar{v}_o = \text{const.}$$

The acceleration is given by eq.(10). Integrating eq.(22) between 0^- and the present time t along the trajectories and setting

$f_1(\bar{x}_o,\bar{v}_o,0^-) = 0$, we get, for $t > 0$,

$$f_1(\bar{x},\bar{v},t) = \&(e/m)\{\int_{0^-}^{t}\delta(y_o+v_y t')\delta(t')dt'\ \partial f_o/\partial v_z + \int_{0^-}^{t}\delta'(y_o+v_y t')\theta(t')\,dt'\,(v_y\partial f_o/\partial v_z - v_z\partial f_o/\partial v_y)\}.$$

These integrals give generalized functions which must be inter

preted as operating on functions of the velocity v_y. Thus,

$$\int_{0^-}^{t} dt' \; \delta(y_o + v_y t')\delta(t') = \delta(y_o) = \delta(v_y - y/t)/t \; ,$$

since $y_o = y - v_y t$. Similarly, integrating by parts,

$$\int_{0^-}^{t} dt' \; \delta'(y_o + v_y t')\theta(t') = [\delta(y_o + v_y t) - \delta(y_o)]/v_y =$$

$$= [\delta(y) - \delta(v_y - y/t)/t]/v_y \; ,$$

when $v_y \neq 0$. Hence, we obtain

$$f_1(\bar{x},\bar{v},t) = \&(e/m)\{\frac{1}{t}\,\delta(v_y - y/t)\;\partial f_o/\partial v_z + \delta(y)\;\partial f_o/\partial v_z -$$

$$- \frac{1}{t}\,\delta(v_y - y/t)\;\partial f_o/\partial v_z -$$

$$- (v_z/v_y)\;[\delta(y) - \frac{1}{t}\,\delta(v_y - y/t)]\;\partial f_o/\partial v_y \qquad (23)$$

where the first and third terms cancel. Since $\sigma = -en_o f_1$, this confirms eq.(11), deduced using physical arguments. For $v_y = 0$, the integration over time gives

$$f_1(\bar{x},v_x,0,v_z,t) = \&(e/m)[\delta(y)\;\partial f_o/\partial v_z - \delta'(y)\;t\;v_z\;\partial f_o/\partial v_y] \; ,$$

which is equivalent to eq.(12). Therefore, the percussion approach can also be regarded as a special case of integration along trajectories. It must be observed, however, that the fields act at y=0. Hence, the motion of the particles is exactly field free for $y \neq 0$ and not as a consequence of a linear approximation.

8. CONCLUSIONS

We obtained the space-time polarization response of plasmas for electromagnetic excitations from physical considerations, without Fourier transforms. The given one dimensional response tensor includes the possibility of beams in the plasma. For applications to specific problems, of course, we must sum the po

larization responses of the different species of plasma particles and the contributions of beams, if any. Plasma electrodynamical problems require the solution of integro-differential equations like eq.(20). Considering the complexity and subtleties of the inversion of Fourier transforms, necessary to calculate Green's tensors for plasmas, particularly in problems with more than one dimension,([6]) a direct attack of eq.(20) by numerical or other approximation methods may be also helpful or complementary in some problems. We may also note that when a limited period of time is under study, there is an advantage in proceeding in this fashion. In the Fourier method, the full extent of the frequency spectrum should be used in the inversion when the temporal behaviour is desired without serious distortion. In practice, only asymptotic, $t \to \infty$, representations of the Green's functions are usually calculated, which are valuable for stability studies but are not sufficient for a complete $\bar{x}$, t description.

The calculation of the three dimensional response tensor for quasi-electrostatic excitations of plasmas in external magnetic fields seems to be a tractable problem. The percussion approach looks promising as well for the study of non linear electrostatic responses in the $\bar{x}$, t description.

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