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A NEW PROOF OF TIRAO'S RESTRICTION THEOREM

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0. Let $\mathfrak{g}_{\mathbf{R}} = \mathfrak{k}_{\mathbf{R}} \oplus \mathfrak{p}_{\mathbf{R}}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}_{\mathbf{R}}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding complexification. Let θ be the associated Cartan involution. Also let $\mathfrak{a}_{\mathbf{R}}$ be a maximal abelian subspace of $\mathfrak{p}_{\mathbf{R}}$ and let $\mathfrak{a}$ be its complexification. Now let G be the adjoint group of $\mathfrak{g}$ and let K be analytic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$. Also let M be the centralizer of $\mathfrak{a}$ in K. If $S'(\mathfrak{g})$ denotes the ring of all polynomial functions on $\mathfrak{g}$ then clearly $S'(\mathfrak{g})$ is a G-module and a fortiori a K-module.

In this paper, we shall give a new proof of the Tirao Restriction Theorem which characterizes the image of $S'(\mathfrak{g})^K$ in the split rank one case (i.e., dim $\mathfrak{a} = 1$) via the monomorphism induced by restriction

$$S'(\mathfrak{g})K \rightarrow S'(\mathfrak{k})^M \oplus S'(\mathfrak{a})$$

The restriction homomorphism $\mathbf{R}$: $S'(\mathfrak{g}) \rightarrow S'(\mathfrak{k}\oplus\mathfrak{a})$ induces a monomorphism $S'(\mathfrak{g})^K \rightarrow S'(\mathfrak{k})^M \otimes S'(\mathfrak{a})$, because $K.(\mathfrak{k} \oplus \mathfrak{a})$ is dense in $\mathfrak{g}$. More than this, in [KT] a suitable element $b \in S'(\mathfrak{g})^K$ is defined and the following theorem of Kostant is proved:

If $b_o = b\,|_{\mathfrak{k} \oplus \mathfrak{a}}$, then the restriction homomorphism $\mathbf{R}$ induces an isomorphism of $S'(\mathfrak{g})^K{}_b$ onto $(S'(\mathfrak{k}\oplus \mathfrak{a}))^{M'}{}_{b_o}$. (Here we use the usual notation: if A is a commutative ring, $b \in A$, $X = \mathrm{Spec}\ A$ or Spec max A, then A_b is the localization of A by b and X_b is the principal open set defined by b).

This theorem is the initial point of the Tirao's proof of his Restriction Theorem [T]. Though our proof does not make use of it, we shall give a slight generalization in 6.

1. As usual, $L^\wedge$ notes the set of equivalence classes of finite dimensional irreducible representations of an algebraic reductive complex linear group. We will confuse $\tau \in L^\wedge$ with the space on which acts. We will exploit the following well known version of the Schur Lemma:

For $\tau, \lambda \in L^\wedge$, $\dim (\tau \otimes \lambda)^L = 1$ if $\tau = \lambda^*$, 0 otherwise.

If V is any L-module, $\tau \in L^\wedge$, note by $\mathcal{V}_\tau$ the isotypic component of type τ. Put $d(\tau) = \dim \tau$.

Now let $Gr(L)$ be the category of complex graded L-modules $M = \oplus_{n \geq 0} M_n$, such that $\dim_{\mathbf{C}} M_n < \infty\ \forall$ n and put $x(M)$ for the Poincaré series of M, i.e. $\Sigma \dim_{\mathbf{C}} M_n t^n \in \mathbf{Z}[[t]]$. As a first consequence of the Schur Lemma, we shall prove:

Lemma: If $M, N \in Gr(L)$ then

$$x((M \otimes N)^L) = \Sigma_{\lambda \in L^\wedge} d(\lambda)^{-2} x(M_\lambda) . x(N_{\lambda^*}).$$

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Proof: $M = \oplus_{\lambda \in L^\wedge} M_\lambda$, $N = \oplus_{\tau \in L^\wedge} N_\tau$ and then

$$(M \otimes N)^L = \oplus_{\lambda,\tau} (M_\lambda \otimes N_\tau)^L = \oplus_{\lambda \in L^\wedge} (M_\lambda \otimes N_{\lambda^*})^L$$

So, $x((M \otimes N)^L) = \Sigma_{\lambda \in L^\wedge} x(M_\lambda \otimes N_{\lambda^*})^L$.

Now $N_{\lambda^*} = \oplus_{s \geq 0} (N_{\lambda^*})_s$ and $M_\lambda = \oplus_{r \geq 0} (M_\lambda)_r$, hence

$$(M_\lambda \otimes N_{\lambda^*})^L = \oplus_{p \geq 0} (\oplus_{r+s=p} ((M_\lambda)_r \otimes (N_{\lambda^*})_s)^L$$

But

$$\dim ((M_\lambda)_r \otimes (N_{\lambda^*})_s)^L = d(\lambda)^{-2} \dim (M_\lambda)_r \dim (N_{\lambda^*})_s$$

and hence

$$x(M_\lambda \otimes N_{\lambda^*})^L = \Sigma_{p \geq 0} (\Sigma_{r+s=p} d(\lambda)^{-2} \dim (M_\lambda)_r \dim (N_{\lambda^*})_s) t^p$$

$$= d(\lambda)^{-2} x(M_\lambda).\, x(N_{\lambda^*}).$$

2. Now let g, a, K, etc, be as in the introduction. Let also K_θ the subgroup of Ad(g) consisting of those elements which commute with θ. From [KR] we know that $S'(p) = S'(p)^K \otimes H$, where H is a homogeneous K_θ-submodule of $S'(p)$. Let

$$T = \{ \tau \in K_\theta^\wedge : \tau^{M_\theta} \neq 0,\ M_\theta = \text{centralizer of } a \text{ in } K_\theta \}\ .$$

We have $H = \oplus_{\tau \in T} H_\tau$ and multiplicity $(\tau, H) = \dim \tau^{M_\theta}$. Assume rank split $g = 1$. Then ([K], 2.2.g) $\dim \tau^{M_\theta} = 1\ \forall \tau \in T$. Now put $\Gamma = \{ \tau \in K^\wedge : \tau^M \neq 0 \}$(note that $\gamma \in \Gamma \Rightarrow \gamma^* \in \Gamma$). It is also clear that $H = \oplus_{\gamma \in \Gamma} H_\gamma$ and hence we have for each $\gamma \in \Gamma$ (see [T]):

1) H_γ is homogeneous of degree, say, $m(\gamma^*)$.

2) $\dim \gamma^M = 1$.

3. Theorem ([T]): *Let g, a, etc, as in the introduction. Assume* $\dim a = 1$. *Let V be a finite dimensional* K-module.

Then the restriction homomorphism $S'(V \oplus p) \to S'(V \oplus a)$ induces an isomorphism of $S'(V \oplus p)^K$ onto

$$(\oplus_{n \geq 0} (\oplus_{\lambda \in \Gamma,\, m(\lambda) \leq n} S'(V)_\lambda{}^M \otimes S'_n(a)))^W.$$

Proof: If E_1, E_2 are finite dimensional L-modules, the later trivial, $(E_1 \otimes E_2)^L = E_1{}^L \otimes E_2$. (For ex., it is easy to see that $\mathrm{Hom}_L(E_2, E_1) = \mathrm{Hom}(E_2, E_1{}^L)$). Thus

$$S'(V \oplus p)^K = (S'(V) \otimes S'(p))^K = (S'(V) \otimes S'(p)^K \otimes H)^K$$

$$= S'(p)^{k} \otimes (S'(\mathcal{V}) \otimes H)^{K}$$
$$= S'(p)^{K} \otimes (\oplus_{\lambda \in \Gamma}(S'(V)_{\lambda} \otimes H_{\lambda^*})^{K}$$
$$\xrightarrow{\sigma} S'(a)^{W} \otimes (\oplus_{\lambda \in \Gamma} \sigma (S'(\mathcal{V})_{\lambda} \otimes H_{\lambda^*})^{K}$$

Let us look more closely at $(S'(V)_{\lambda} \otimes H_{\lambda^*})^{K}$. In general, if $S'(V)_{\lambda} = \oplus_i W_i$, $W_i \cong \lambda$ and $H_{\lambda^*} = \oplus_j U_j$, $U_j \cong \lambda^*$, then $(S'(V)_{\lambda} \otimes H_{\lambda^*})^{K} = \oplus_{i,j} (W_i \otimes U_j)^{k}$.

But σ is 1-1 and $\dim (W_i \otimes U_j)^{K} = \dim (W_i)^{M} = 1$. So

$$\sigma(S'(V)_{\lambda} \otimes H_{\lambda^*})^{K} = S'(V)^{M}_{\lambda} \otimes S'_{m(\lambda)}(a)$$

Hence

$$\sigma(S'(V \oplus p)^{K}) = S'(a)^{W} \otimes (\oplus_{\lambda \in \Gamma} S'(V)_{\lambda}^{M} \otimes S'_{m(\lambda)}(a$$

Therefore, we have:

$$s(S'(V \oplus p)^{k}) =$$

$$(\oplus_{n \geq 0} S_n'(a) \otimes (\oplus_{\lambda \in \Gamma} S'(V)_{\lambda}^{M} \otimes S'_{m(\lambda)}(a)))^{W}$$

$$=(\oplus_{n \geq 0} (\oplus_{\lambda \in \Gamma, m(\lambda) \leq n} S'(V)_{\lambda}^{M} \otimes S'_{n}(a)))^{W}.$$

4. **Remarks.** The proof in [T] is more geometric and uses the curves $t^m\mu(t)v$, for $t \in C^x$, $v \in a - 0$, and $m \in \mathbf{Z}$, μ: $C^x \to K$ a one parameter subgroup suitably choosen.

Now let $f \in S'(V)^{M}$, $f = \Sigma f_{\lambda}$, $f_{\lambda} \in S'(V)_{\lambda}^{M}$. Put

$$d(f) = \sup \{ m(\lambda): f_{\lambda} \neq 0 \}.$$

This function, introduced by Tirao, gives in fact an ascending filtration in $S'(V)^{M} = C$, with

$$C_j = \{ f \in S'(V)^{M}: d(f) \leq j \} = \oplus_{\gamma \in \Gamma, m(\gamma) \leq j} S'(V)_{\gamma}^{M}.$$

Moreover, Tirao observed that $d(fg) = d(f) + d(g)$; that is, gr C is a domain. On the other hand, we shall make the following observation: put

$$D = \oplus_{n \geq 0} (\oplus_{\lambda \in \Gamma, m(\lambda) \leq n} S'(V)_{\lambda}^{M} \otimes S_n'(a)).$$

Let $H \in a^* - 0$. Then it is not hard to see that $D / D.H \cong \mathrm{gr}\, C$. As H is homogeneous in the usual grading, note that gr C has two graded structures: the usual one and the given by d.

Finally, if $V = k$ and rank g = rank k, it is known that W acts trivially on V. So in this case the Theorem can be rewritten:

$$\sigma(S'(g)^{K}) = S'(a)^{W} \otimes (\oplus_{\lambda \in \Gamma} S'(k)_{\lambda}^{M} \otimes S'_{m(\lambda)}(a)),$$
$$= \oplus_{n \geq 0} (\oplus_{\lambda \in \Gamma, m(\lambda) \leq 2n} S'(k)_{\lambda}^{M} \otimes S'_{2n}(a))$$

because $S'_{m(\lambda)}(a) \subseteq S'(a)^{W}$. Hence, in this case

$$\sigma(S'(\mathfrak{g})^K) / \sigma(S'(\mathfrak{g})^K).\mathrm{H}^2 \cong \mathrm{gr}\,\mathrm{C},$$

modulo an adequate renumbering of the filtration.

5. Let L be a connected reductive algebraic group over an algebraically closed field k of char 0, and let X be an affine algebraic k-variety on which L acts morphically (briefly, a L-variety). Note by $k[X]$ the ring of regular function on X, X/L the affine variety associated to $k[X]^L$ and $\Pi = \Pi_X : X \to X/L$ the morphism associated to the inclusion. We say that X has generically closed orbits if the union of all closed orbits contains an open dense subset of X. Between the isotropy subgroups of points in closed orbits, there is one whose conjugacy class is minimal; we say that it is of principal type (see e.g. [LR]). Also we set as usual $N_L(H)$ for the normalizer in L of a subgroup H.

6. **Theorem**: *Let X_1, X_2 be irreducible normal affine L-varieties. Assume that X_2 has generically closed orbits and pick* a $\in X_2$ *in the interior of the union of all closed orbits* and *such that the stabilizer $L^{a} = H$ is of principal type. Furthermore, assume that X_2^H is irreducible.*

Then there exists b $\in k[X_2]^L$ *such that the restriction*

$$k[X_1 \times X_2] \to k[X_1 \times X_2^H]$$

induces an isomorphism

$$k[(X_1 \times X_2)_{b}]^L \to k[(X_1 \times X_2^H)_{b_0}]^{N_L(H)},$$

that is

$$k[X_1 \times X_2]^L_{b} \to k[X_1 \times X_2^H]_{b_0}^{N_L(H)}$$

where b_0 *is the restriction of* b.

Proof: We will use the same tools as in [LR]. So we recall:
i) ([L R], Lemma 2.1). Let $\varphi : X \to Y$ be a surjective birational morphism of irreducible affine algebraic k-varieties. If Y is normal, then φ is an isomorphism of varieties.
ii) ([LR], Th. 2.2). Let the reductive algebraic group M act on the irreducible normal affine variety X. Let C be a closed reductive subgroup of M and let E be a closed C-stable subvariety of X such that the following conditions hold:

C 1. E/C is irreducible.

C 2. Each closed M-orbit on X meets E.

C 3. There exists a non-empty open subset V of X/M such that for every $\xi \in V$, the intersection $\pi^{-1}(\xi) \cap E$ contains a unique closed C-orbit.

Then the restriction homomorphism $k[X] \to k[E]$ maps $k[X]^M$ isomorphically onto $k[E]^{N_M(H)}$.

Now, from [R] and the hypothesis on X_2, there exists a non-empty L-stable open subset U of X_2 containing a and satisfying: if x $\in U$, Lx is closed and L^x is of principal type.

From the fact that the elements in $k[X_2]^L$ separate disjoint closed L-invariants subsets, it follows the existence of $b \in k[X_2]^L$ such that $b(a) \neq 0$, $(X_2)_b \subseteq U$. Denote also by b the extension to $X_1 \times X_2$.

Now we want to apply ii) to $X = (X_1 \times X_2)_b$, $E = (X_1 \times X_2{}^H)_{b_0}$,and

$C = N_L(H)$ (see [LR] for the reductivity). C1 follows immediately from the hypothesis. If $(x_1, x_2) \in X$, Lx_2 is closed and then C2 follows from [LR], Lemma 3.5. We claim that the intersection of a closed L-orbit on X with E is a closed $N_L(H)$-orbit. Let (x,y) M$\in E$, $g \in LD$ *such that* $(gx, gy) \in E$; thus $y, gy \in (X_2{}^H)_{b_0}$, so $L^y = H = L^{gy} = gHg^{-1}$; then $g \in N_L(H)$.

The uniqueness in C3 and hence the proposition will follow from the following lemma.

Lemma: *Keep the notations and the hypothesis as above. If* $z \in \mathcal{E}$ *and* $N_L(H)z$ *is closed in E, then* Lz *is closed in X.*

Proof: Let $\zeta: L \times E \to X$ induced by the action and let $N_L(H)$ act on $L \times E$ via $u(g,z) = (gu^{-1}, uz)$. We claim omitting the straightforward proofs:

a) $\forall z \in \mathcal{E}, \zeta^{-1}(Lz) = L \times N_L(H)z$

b) If $(g_i, z_i) \in L \times E$ $(i = 1,2)$ then,

$\zeta(g_1, z_1) = \zeta(g_2, z_2)$ iff $(g_1, z_1) \in N_L(H)(g_2, z_2)$.

Let us consider

$$
\begin{array}{ccc}
L \times E & \xrightarrow{\xi} & X \\
\pi & & \nu \\
 & & \\
(L \times E) / N_L(H) & &
\end{array}
$$

It follows from b) the existence of ν and clearly, it is bijective; since we are in char. 0, ν is birational. Now $L \times E$ is an irreducible affine variety (L is connected!) and X is normal by hypothesis. Then we get from i) that ν is an isomorphism.

On the other hand, π maps closed C-invariant subsets of $L \times E$ on closed subsets of $(L \times E)/C$ (see e.g. [M], p.28). Hence Lz is closed if and only if $L \times N_L(H)z$ is.

7. **Remark:** The generic closedness is not a superfluous condition, as the following example shows. Let $L = SL(2,\mathbb{C})$, $V_1 = V_2$ $\mathbb{C}^2$ with the natural action. The unique closed orbit in V_2 is 0; hence the principal isotropy group is L. Therefore,

$$S'(V_1 \oplus V_2{}^H)^{N_L(H)} = S'(V_1)^L = \mathbb{C},$$

but

$$S'(V_1 \oplus V_2)^L = \mathbb{C}\,[\det].$$

References

[K] KOSTANT, B . *On the existence and irreducibility of certain series of representations,* Lie Groups and their Representations, 1971 Summer School in Math. (I.M. Gelfand, ed.), Wiley, New York, 1975, pp. 231-330.

[KR] KOSTANT, B and RALLIS, S. *Orbits and representations associated with symmetric spaces.* Amer. J. of Math. 93 (1971), pp. 753-809.

[KT] KOSTANT, B. and TIRAO, J. *On the structure of certain subalgebras of a universal envelopping algebra.* Trans. A.M.S. 218 (1976), pp.133-154.

[LR] LUNA, D and RICHARDSON, R.W. *A generalization of the Chevalley Restriction Theorem.* Duke J. of Math. 46 (1979), pp. 487-497.

[M] MUMFORD, D. *Geometric Invariant Theory.* Ergebnisse Band 34. Berlin: Springer Verlag.

[R] RICHARDSON, R.W. *Principal Orbit Types for Algebraic Transformation spaces in characteristic zero.* Inventiones Math. 16 (1972), 6-15.

[T] TIRAO, J. *A Restriction Theorem for Semisimple Lie Groups of Rank One.* Trans. A.M.S 279 (1983), pp. 651-660.

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