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ON NATURAL AUTOMORPHISMS OF A JOIN OF GRAPHS

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Abstract. Hemminger gives necessary and sufficient conditions for the group of automorphisms of an X-join Z of graphs to consist precisely of natural ones. In this paper we state other conditions in terms of externally related subsets in Z. We present the lattice of partitions P of the vertex set of a graph X such that every automorphism of X is natural with respect to P, which gives another approach to this problem.

1. Introduction

In this paper we only consider undirected graphs (finite or infinite) without any loops or multiple edges. For undefined terminology see [10].

For a given graph X, we denote V(X) the vertex set (or X if no confusion arises), E(X) the edge set, [x,y] the edge joining vertices x and y, X[A] the subgraph induced by $A \subseteq V(X)$ and G(X) the group of graph automorphisms of X.

Let A and B be subsets of V(X); *A properly cuts* B if $A \cap B$, A-B and B-A are not empty.

A subset A of V(X) is *externally related* in X if for any x,w in A and z in X-A, $[x,z] \in E(X)$ if and only if $[w,z] \in E(X)$.

Let S(A) be the permutation group of a set A. If G and H are subgroups of S(A) and S(B) respectively, the application s of $G \times H^A$ in S(AxB) such that $s(g,(h_x)_{x \in A})(a,b) = (g(a),h_a(b))$ is an injective homomorphism of groups. The image of s is a subgroup of S(AxB), isomorphic to GxH^A, called wreath product of G and H [9]. It will be denoted G_0H. A special case of wreath product is investigated in [5].

For any $x \in X$ let Y_x be a graph. The *X-join* of the family $(Y_x)_{x \in X}$ is the graph Z with

$$V(Z) = \{(x,y) : x \in X, y \in Y_x\}$$

and

$$E(Z)=\{[(x,y),(x',y')] : [x,x'] \in E(X) \text{ or else } x = x' \text{ and } [y,y'] \in E(Y_x)\}$$

The lexicographic product of graphs X and Y (or composition), denoted XoY, is the X-join of the family $(Y_x)_{x \in X}$ with $Y_x = Y$, for each $x \in X$.

The lexicographic product of two graphs was introduced by Harary in [11] with the purpose of constructing a binary operation on finite graphs such that the automorphism group of the product was the wreath product of the automorphism groups of the composants.

Sabidussi [16,17] proved, for certain classes of graphs, not necessarily finite, that $G(X)_0G(Y) = G(XoY)$ if and only if

(1) Y is connected if $R \neq I$,

(2) the complement of Y is connected if $S \neq I$,

where R and S are the following equivalence relations on V(X)

xRy if and only if {x,y} is externally related in X and $[x,y] \notin E(X)$,

xSy if and only if $x = y$ or {x, y} is externally related in X and $[x, y] \in E(X)$, and I is the identity relation on X.

In [12,13] Hemminger extended this result by enlarging the class of graphs under consideration.

In [18] Sabidussi introduced the X-join of graphs and Hemminger [14] generalized the problem of Harary for a graph Z, X-join of $(Y_x)_{x \in X}$, where

(3) $Y_x \cong Y_w$ if $f(x) = w$ for some $f \in G(X)$.

He found necessary and sufficient conditions for any automorphism h of Z to be of the form $h(x,y) = (f(x), g_x(y))$ with $f \in G(X)$ and g_x an isomorphism of Y_x onto $Y_{f(x)}$, for any $x \in X$. These automorphisms are called *natural.*

Hemminger's theorem was generalized by Dörfler and Imrich [2] for hypergraphs without loops and by Dörfler [3] for directed graphs.

Independently, Hahn [7] generalized the result of Sabidussi for the lexicographic product of two hypergraphs H and H' with H' finite and [8] for the lexicographic product of two directed graphs D and D' with D' finite.

In all these generalizations Sabidussi's conditions (1) and (2) are maintained.

The purpose of sections 2 and 3 of this paper is to show the relationship between natural automorphisms of an X-join Z of $(Y_x)_{x \in X}$ and the externally related subsets of Z. Theorem 3.1. gives another viewpoint to Theorem 2.10 [14] that also replaces the conditions (3) by a milder one. It is shown (3.2) that Sabidussi's conditions (1) and (2) "mean" that none of the externally related subsets of Z properly cuts two graphs of the family $(Y_x)_{x \in X}$. The principal tool is the characterization of externally related subsets in a join (Ths. 2.1, 2.2.and 2.3).

In section 4 we see Hemminger's problem from another point of view.

Let X be a graph. The set of partitions P of V(X) such that each block of P is externally related in X is a sublattice T(X) [6] of $\Pi(X)$, the lattice of all the partitions of V(X).

If $P \in T(X)$ and X/P is the quotient graph then X is isomorphic to the X/P-join of $(X[B])_{B \in P}$. We introduce a sublattice $\Sigma(X)$ of $\Pi(X)$ such that $P \in \Sigma(X) \cap T(X)$ if and only if P decomposes X as a join with the property that each automorphism of X is natural. The automorphisms of X are then induced by automorphisms of X/P. We show some distinguished elements of $\Sigma(X) \cap T(X)$.

2. Externally related sets in a graph

Let X be a graph, A and B subsets of V(X). A direct verification shows that, if A and B are externally related in X, then $A \cap B$ is externally related too; if, in addition, A properly cuts B then $A \cup B$, $A \Delta B$(symmetric difference) and A-B are also externally related.

It is easy to see that if (A,B) is a partition of V(X) such that A and B are both externally related in X then there are only two possibilities: for any $a \in A$ and $b \in B$, $[a,b] \in E(X)$, or for any $a \in A$ and $b \in B$, $[a,b] \notin E(X)$. It is said in [1] that X is of *type 1*, in the first case and of *type 0* in the second.

In [4] are defined the restricted homogeneous subsets in an hypergraph which generalize the externally related subsets in a graph. As a corollary of the results in [1], extended for infinite hypergraphs, we have the following characterization of the externally related subsets in a join. But first some definitions.

Let Z be the X-*join* of $(Y_x)_{x \in X}$. For $W \subseteq V(Z)$ let be

$$\bar{W} = \{x \in X; W \cap Y_x \neq \emptyset\}.$$

If $A \subseteq V(X)$ and $x \in A$ then, x *is of type 1* (respectively 0) in A if for any $y \in A-x$, $[x,y] \in E(X)$ (resp. $[x,y] \notin E(X)$).

2.1. Theorem. *If* $W \subseteq Y_x$ *for some* $x \in X$ *then* W *is externally-related in* Z *if and only if* $\tilde{W}$ *is externally related in* Y_x.

2.2. Theorem. *If* $W = \cup_{x \in A} Y_x$ *for some* $A \subseteq V(X)$ *then* W *is externally related in* Z *if and only if* W *is externally related in* X.

2.3. Theorem. *Let* $W \subseteq V(Z)$ *such that* W *properly cuts* Y_a *for some* $a \in X$. *Then* W *is externally related in* Z *if and only if*

a) $\overline{W}$ *is externally related in* X

b) *For any* $x \in X$, $W \cap Y_x$ *and* $Y_x - W$ *are externally related in* Y_x.

c) *There is* t, t=0 *or* t=1, *such that for any* $x \in X$, *if* W *properly cuts* Y_x *then* x *is of type* t *in* $\overline{W}$ *and* Y_x *is of type* t.

The set of all the partitions of V(X) forms a complete lattice $\Pi(X)$ [15]. The unit element is $\{V(X)\}$; the others are called proper partitions. The least upper bound (resp. greatest lower bound) of the partitions P_1 and P_2 will be denoted $P_1 \vee P_2$ (resp. $P_1 \wedge P_2$).

The partitions whose blocks are externally related in X form a complete sublattice T(X) of $\Pi(X)$ [6].

If P is a partition of V(X) we define the *quotient graph* of X by P, denoted X/P, by $V(X/P)=P$ and $E(X/P)=\{[B,B'];B,B' \in P, B \neq B'$, there is $b \in B$ and $b' \in B'$ with $[b,b'] \in E(X)\}$.

Let X_i be a graph, i=1,2. If h is a graph isomorphism of X_1 onto X_2 then we define a mapping h^* of $\Pi(X_1)$ onto $\Pi(X_2)$ by: for any P_1 8M« P (X_1), $h^*(P_1) = \{h(B); B \in P_1\}$. It follows that h^* is a lattice isomorphism. Furthermore, h^* preserves the least upper bound and the greatest lower bound of any family. The restriction of h^* to $T(X_1)$ is a lattice isomorphism onto $T(X_2)$.

Let $P_i \in \Pi(X_i)$, i=1,2. If $h^*(P_1)=P_2$, then h induces an isomorphism of X_1/P_1 onto X_2/P_2.

3. Unnatural automorphisms of an X-join

Let Z be the X-join of $(Y_x)_{x \in X}$. A *subjoin* of Z is a X[A]-join of $(Y_x)_{x \in A}$ for some $A \subseteq V(X)$.

We call $\{V(Y_x); x \in X\}$ the *canonical partition* of Z.

Let Z_i be the X_i-join of $(Y_{ix})_{x \in X}$, i=1,2, and h a graph isomorphism of Z_1 onto Z_2. Hemminger [14] calls h natural if for each $x_1 \in X_1$ there is an $x_2 \in X_2$ such that $h(Y_{1x_1})=Y_{2x_2}$, that is h^* assigns to the canonical partition of X_1 the canonical partition of X_2.

3.1 Theorem. *Let* Z *be the* X-join *of* $(Y_x)_{x \in X}$ *such that if* $\{x,w\}$ *is externally related in* X *then* $Y_x \cong Y_w$. *Then* G(Z) *contains an unnatural automorphism if and only if one of the following conditions holds.*

(i) *There is an externally related subset in* Z *that properly cuts two blocks of the canonical partition of* Z.

(ii) *There is an externally related subset* W *in* Z *such that, for some* $a \in X$, W *properly cuts* Y_a *and* $Z[W] \cong Y_a$.

(iii) *There are two proper partitions in* T(Z), P_1 *and* P_2, *greater than the canonical partition of* Z *and an isomorphism* s *of* Z/P_1 *onto* Z/P_2 *such that, for any block* B *of* P_1, *the subjoins of* Z, Z[B] *and* Z[s(B)], *are isomorphic and, for at least one* B, *there exists an unnatural isomorphism.*

Proof. Let h be an unnatural automorphism of Z and P the canonical partition of Z. Assume that (i) and (ii) do not hold.

Let $P_1=P \vee (h^{-1})^*(P)$ and $P_2=P \vee h^*(P)$. Then $h^*(P_1)=P_2$ and P_1,P_2 are both in T(Z).

Assume that P_2 is not a proper partition. Because h is unnatural, $|X| > 1$, then P and h*(P) are proper partitions; thus P and h*(P) are not comparable in T(Z). Since (i) is not true, there must be x,w in X such that Yx properly cuts $h(Y_w)$ and $Y_x \cup h(Y_w) = V(Z)$. Hence $y \neq x$ implies $Y_y \subset h(Y_w)$ and $y \neq w$ implies $h(Y_y) \subset Y_x$. In addition, since (ii) does not hold, $x \neq w$.

The next step is to show that {x,w} is externally related in X.

Let z be an element of X different from x and w and let [z,w] be an edge of X. Then, for any $a \in Y_z$, $b \in Y_w$: $[a,b] \in E(Z)$. Since h is an automorphism, $[h(a), h(b)] \in E(Z)$; we can choose b such that h(b) is in Yz. As we have noted, $h(Y_z) \subset Y_x$ then [h(a),h(b)] is an edge between Y_x and Y_z. Therefore, $[x,z] \in E(X)$. By taking $a \in Y_z$, $b \in Y_x \cap h(Y_z)$ and h^{-1}, we can see that $[z,x] \in E(Z)$ implies $[z,w] \in E(Z)$.

By the condition of the hypothesis, $Y_x \cong Y_w$, which means that (ii) holds. This is a contradiction; thus P_2, and therefore P_1, are proper partitions.

Let s be the isomorphism of Z/P_1 onto Z/P_2 induced by h. For any block B of P_1 the restriction of h is an isomorphism of Z[B] onto Z[s(B)]. Since h is unnatural, at least one of these isomorphisms must be unnatural. So condition (iii) is satisfied.

Conversely, let us prove that one of the above conditions implies the existence of an unnatural automorphism.

Assume that (i) is true. Let W be an externally related subset in Z which properly cuts Ya,Yb with $a \neq b$. We can suppose that W intersects only Y_a, Y_b because $W' = (W - Y_a) \Delta (W - Y_b)$ is externally related in Z and intersects only Y_a, Y_b. By Theorem 2.3, {a,b} is externally related in X then $Y_a \cong Y_b$.

Let f be an isomorphism of Ya onto Yb. We use Sabidussi's construction [16, p.694] and Theorem 2.3 for defining the following unnatural automorphism h of Z: h coincides with f on $W \cap Y_a$, with f^{-1} on $f(W \cap Y_a)$ and with the identity on the remainder of the graph.

Assume that (ii) is true. Let W be an externally related subset in Z, such that Z[W] is isomorphic to Y_a and W properly cuts Y_a.

Let f be an isomorphism of Z[W] onto Y_a. We can assume that W properly cuts only Y_a. By using Hemminger's construction [12, p.500] and Theorem 2.3 we define the following unnatural automorphism h of Z: h coincides with the identity on $f(W \cap Y_a) \cup U_{x \notin \bar{W}} Y_x$ and, for any $x \in \bar{W}$ {a}, h coincides with f on Y_x and with f^{-1} on $f(Y_x)$.

Assume that (iii) is true. For each block B of P_1, choose an automorphism h_B of Z[B] onto Z[s(B)] such that at least one of these automorphisms is unnatural. The mapping h which coincides whith h_B on each block B of P_1 is evidently an unnatural automorphism of Z.

3.2 Remarks. 1) We show the relationship between Hemminger's Theorem 2.10 [14] and Theorem 3.1. First, we call (h) the initial hypothesis of Theorem 3.1: "if {x,w} is externally related in X then $Y_x \cong Y_w$".

a) The initial hypothesis of Theorem 2.10 "$Y_x \cong Y_w$ if $u(x) = w$ for some $u \in G(X)$" is evidently stronger than (h).

The conditions (1),...,(4) of Theorem 2.10 and (i),(ii),(iii) are related in the following manner.

b) The condition (i) implies that (1) or (2) (Sabidussi's conditions) does not hold. Conversely, if (h) is true but (1) or (2) is not then (i) is verified.

Indeed, if (i) is true then there exists an externally related subset W in Z which properly cuts two blocks, Y_x and Y_w, and does not intersect any other. By Theorem 2.3,{x,w} is externally related in X and if $[x,w] \notin E(X)$ then $(x,w) \in R$ and x is of type 0 in $\bar{W}$. Hence, Y_x is of type 0 and so Y_x is not connected. In the same way, if [x,w] belongs to E(X) then (2) does not hold.

Conversely, suppose (h) and the existence of two vertices x,w related by R such that Y_x is not connected. Since {x,w} is externally related in X, $Y_x \cong Y_w$. The union of a component of Y_x and a component of Y_w satisfies (i). The same result is obtained if (2) does not hold.

c) Let (g) be the following condition: "there is an externally related subset W in Z such that, for some $x \in X$, W properly cuts Y_x, $Z[W] \cong Y_x$ and, for any $w \neq x$, $W \supset Y_w$". It is easy to construct an example which shows that (g) is stronger than (ii). It follows from Theorem 2.3 that (g) is equivalent to the negation of (4).

d) The condition (iii) is equivalent to the negation of (3).

2) In theorem 2-21 [14] Hemminger removes the initial hypothesis of Theorem 2-10 (cf. remark 1)), but the following example shows that the first part of Theorem 2.21 has an error.

Let X, Y_i, $i=1,2,3$, be the graphs of Fig.1 and Z be the X-join of Y_1, Y_2 and Y_3.

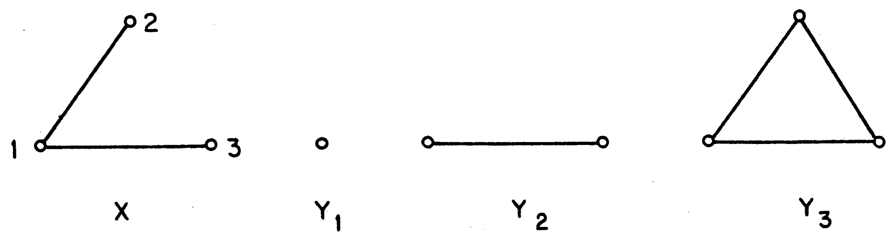

Fig.1

Clearly, G(Z) consists only of natural automorphisms but (2.3) belongs to R and Y_2 is not isomorphic to Y_3.

3) It is easy to generalize Theorem 3.1 for hypergraphs without loops.

Theorem 3.1 of [14] can be formulated in the following manner.

3.2 Corollary. *Let* X *and* Y *be graphs such that* $Y \not\equiv X_1 \circ Y$ *for some subgraph* X_1 *of* X *with* $|X_1| > 1$. Then $G(X \circ Y) = G(X) \circ G(Y)$ *if and only if the externally related subsets in* $X \circ Y$ *isomorphic to* Y *do not properly cut any blocks of the canonical partition and those which are not isomorphic to* Y *properly cut at most one block.*

4. The lattice $\Sigma(X)$

Let X be a graph and $h \in G(X)$. As it is noted above in 2, h^* is a lattice automorphism of $\Pi(X)$.

We denote $\Sigma(X)$ the set of partitions $P \in \Pi(X)$ such that $h^*(P)=P$.

Theorems 4.1 and 4.2 below are easily verifiable.

4.1. Theorem. $\Sigma(X)$ *is a complete sublattice of* $\Pi(X)$.

If L is a sublattice of $\Pi(X)$ and $P \in L$ we denote $[P,1]_L$ the set of partitions in L greater or equal than P. We have $[P,1]_{\Pi(X)} \cong \Pi(X/P)$ [15] and $[P,1]_{T(X)} \cong T(X/P)$ [6].

The analogue of these properties for $\Sigma(X)$ is the following.

4.2 Theorem *If* $P \in \Sigma(X)$ *and for any* $\phi \in G(X/P)$ *there exists* $f \in G(X)$ *such that f induces* ϕ *then* $[P,1]_{\Sigma(X)} \cong \Sigma(X/P)$.

If $P \in T(X)$ then X is the (X/P)-join of the family $(X[B])_{B \in P}$.

Evidently P ∈ Σ(X) if and only if all the automorphisms of X are natural. The problem studied in [14,Th.2.21] is the characterization of the lattice Σ(X) ∩ T(X).

Next, we show that distinguished partitions defined by Sabidussi and Habib are in Σ(X) ∩ T(X).

Sabidussi [16] first introduced the equivalence relations R and S on V(X) (cf. Introduction). It is easy to see that R ∪ S is also an equivalence relation. Let P_R, P_S and $P_{R\cup S}$ be the partitions of V(X) associated with these relations. We have, $P_{R\cup S} = P_R \vee P_S$.

4.3. Theorem: *The partitions* P_R, P_S *and* $P_{R\cup S}$ *are in* Σ(X) ∩ T(X).

Proof: It is clear that B ∈ P_R if and only if B is externally related in X, X[B] is a null graph and B is maximal with these two properties. Similarly, B ∈ P_S if and only if B is externally related in X, X[B] is a complete graph and B is maximal with these two properties. All these properties are preserved under automorphisms.

Let be P_0 the least upper bound of the atoms of T(X) and P_1 the greatest lower bound of the coatoms of T(X), if there exist atoms and coatoms. These partitions were introduced by Habib [6] for finite directed graphs.

If h belongs to G(X), then the properties of h* mentioned in 2 yield the next theorem.

4.4. Theorem. *The partitions* P_0 *and* P_1 *are in* Σ(X) ∩ T(X).

A graph X is called irreducible [6] if the externally related subsets in X are V(X), a vertex or ∅.

The results about P_0 and P_1 of [6] are valid also for infinite graphs:

A subset B of V(X) is a block of P_0 if and only if B is externally related in X and X[B] is an irreducible graph or a maximal null graph or a maximal complete graph.

If X or its complement X^c is not connected then P_1 is the partition of the connected components of X or X^c. In the other case P_1 is the unique coatom of T(X).

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