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NOTES ON GENERALIZED N-LATTICES

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Abstract. In this note we begin the study of the class of generalized N-lattices (G.N-L). We prove some rules valid in these algebras and we show that the class of deductively semisimple generalized N-lattices is the class of modal tetravalued algebras.

1.Introduction

1.1.The notion of N-lattice introduced by H.Rasiowa [20] (See also [3,4,11,12]) corresponds to the algebraic counterpart of the constructive logic with strong negation considered by D.Nelson [19] and A. Markov [8].

A. Monteiro and D.Brignole [3,4] have given a characterization of N-lattices as algebras $(A,1,\sim,\wedge,\vee,\rightarrow)$ of type (0,1,2,2,2) verifying the axioms:

N1) $x \vee 1 = 1$
N2) $x \wedge (x \vee y) = x$.
N3) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$
N4) $\sim\sim x = x$
N5) $\sim(x \wedge y) = \sim x \vee \sim y$
N6) $(\sim x \wedge x) \wedge (\sim y \vee y) = \sim x \wedge x$
B1) $x \rightarrow x = 1$
B2) $(x \rightarrow y) \wedge (\sim x \vee y) = \sim x \vee y$
B3) $x \wedge (x \rightarrow y) = x \wedge (\sim x \vee y)$
B4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
B5) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$

From the axioms N1, N2 and N3 it follows that the system $(A,1,\wedge,\vee)$ is a distributive lattice with greatest element 1 [21]. From the axioms N1 to N5, the system $(A,1,\sim,\wedge,\vee)$ is a De Morgan algebra, and from N6, it is a linear or normal De Morgan algebra. Then $0 = \sim 1$ is the least element of the lattice A ([2,5,9,10]), if we write $x \leq y$ when $x \wedge y = x$.

Several characterizations of semisimple N-lattices have been obtained by A. Monteiro in [15] (See also [12]), and in [13] he proved that they coincide with three-valued Lukasiewicz algebras.

This class of N-lattices has also been studied by D.Vakarelov [22] under the name of classical N-lattices.

1.2. The notion of modal tetravalued algebras was introduced by A.Monteiro, and I.Loureiro [6] characterized them as algebras $(A,1,\sim,\wedge,\vee,\nabla)$ of type (0,1,2,2,1) such that $(A,1,\sim,\wedge,\vee)$ is a De Morgan algebra and verifies the conditions:

M1) $\sim x \vee \nabla x = 1$

M2) $x \wedge \sim x = \sim x \wedge \nabla x$

If in addition the condition L) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$ is satisfied, we have a three-valued Lukasiewicz algebra [6].

The following rules hold in every modal tetravalued algebra:

M3) $x \leq \nabla x$ [7]

M4) $\sim \nabla \sim x \leq x$ [7]

M5) $\sim \nabla x$ is the Boolean complement of ∇x [7]

M6) $\nabla \nabla x = \nabla x$ [6]

M7) $\nabla(x \vee y) = \nabla x \vee \nabla y$ [6]

M8) If $x \leq y$ then $\nabla x \leq \nabla y$

M9) $\nabla(x \wedge y) \leq \nabla x \wedge \nabla y$

M10) $\nabla \sim \nabla \sim x = \sim \nabla \sim x$

M11) $\sim x \wedge x = x \wedge \nabla \sim x$

The same authors [6] considered an implication operation $\rightarrow$ defined on A by means of the formula:

I) $x \rightarrow y = \nabla \sim x \vee y$ [6]

Then

I1) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$ [6]

I2) $x \rightarrow (y \rightarrow x) = 1$

I3) If $1 \rightarrow x = 1$ then $x = 1$ [6]

I4) $((x \rightarrow y) \rightarrow x) \rightarrow x = 1$ [6]

I5) $x \rightarrow x = 1$

2. Generalized N-lattices

2.1. Definition: A generalized N-lattices (G.N-L.) is an algebra $A=(A,1,\sim,\wedge,\vee,\rightarrow)$ of type (0,1,2,2,2) verifying axioms N1 to N5 and B1 to B5 of 1.1.

Then an N-lattice is a G.N-L. which verifies the Kleene inequality N6 of 1.1.

The simplest example of a G.N-L. which is not N-lattice is $(T, 1, \sim, \wedge, \vee, \rightarrow)$ where $(T,1,\sim, \wedge, \vee)$ is the De Morgan algebra of figure 1 and $\rightarrow$ is given by the table 1 :

	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	1	1	1	1
1	0	a	b	a

table 1

$1 = \sim 0$

$a = \sim a$ $b = \sim b$

$0 = \sim 1$

figure 1

2.3. Theorem: In every G.N-L. A the following properties hold:

B6) $1 \rightarrow x = x$

B7) $x \rightarrow 1 = 1$

B8) $x \rightarrow (y \rightarrow x) = 1$

B9) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

B10) $x \rightarrow (x \rightarrow y) = x \rightarrow y$

B11) $y \leq x \rightarrow y$

B12) If $y \leq z$ then $x \to y \leq x \to z$
B13) If $x \leq y$ then $y \to z \leq x \to z$
B14) $(x \to y) \to (x \to z) \leq xM \to (y \to z)$
B15) $\sim x \to (x \to y) = 1$
16) If $x \leq y$ then $x \to y = 1$ and $\sim (x \wedge y) \to \sim x = 1$
B17) If $x \to y = 1$ and $\sim (x \wedge y) \to \sim x = 1$ then $x \leq y$
B18) If $x \wedge z \leq \sim x \vee y$ then $z \leq x \to y$
B19) $x \to (y \to z) = (x \to y) \to (x \to z)$
B20) $(x \vee y) \to z \leq (x \to z) \wedge (y \to z)$
B21) $x \to (y \vee z) \leq (x \to y) \vee (x \to z)$

Proof. We check only B17 and B19

B17: By hypothesis $x \to y = 1$ (1), $\sim (x \wedge y) \to \sim x = 1$ (2). From B3 and (1) $x \leq \sim x \vee y$ (3). From (3), and N4, $x \wedge \sim y \leq \sim x$ (4). On the other hand, from (2), B3 and N4, $\sim (x \wedge y) = \sim (x \wedge y) \wedge ((x \wedge y) \vee \sim x)$ (5). From (5), and N5, $\sim x \vee \sim y = (\sim x \vee \sim y) \wedge (x \vee \sim x) \wedge (y \vee \sim x)$ (6). From (6), taking into account N2, $\sim y = \sim y \wedge (x \vee \sim x) \wedge (y \vee \sim x)$ (7). From (7), (3) and N2, $\sim y = ((x \wedge \sim x) \vee (\sim y \wedge \sim x)) \wedge (y \vee \sim x) \leq (\sim x \vee (\sim y \wedge \sim x)) \wedge (y \vee \sim x) = \sim x \wedge (y \vee \sim x) = \sim x$, then $x \leq y$.

B19: Let $\mu = x \to (y \to z)$, $a = x \wedge (x \to y)$, $b = z$. Then $\mu \wedge a = (x \to (y \to z)) \wedge x \wedge (x \to y) =$

B3) $x \wedge (\sim x \vee (y \to z)) \wedge (x \to y) =$ B3) $x \wedge (\sim x \vee y) \wedge (\sim x \vee (y \to z)) = x \wedge (\sim x \vee (y \wedge (y \to z))) =$ B3)

$x \wedge (\sim x \vee (y \wedge (\sim y \vee z))) \leq x \wedge (\sim x \vee \sim y \vee z) =$ N3) $(x \wedge \sim x) \vee (x \wedge \sim y) \vee (x \wedge z) \leq \sim x \vee (x \wedge \sim y) \vee z$ (1).

On the other hand $\sim a \vee b = \sim (x \wedge (x \to y)) \vee z =$ B3) $\sim (x \wedge (\sim x \vee y)) \vee z =$ N5) $\sim x \vee \sim (\sim x \vee y) \vee z =$ N4) $\sim x \vee (x \wedge \sim y) \vee z$ (2). From (1) and (2) $\mu \wedge a \leq_{\sim} a \vee b$, then by B18 $\mu \leq a \to b$(3), that is $x \to (y \to z) \leq (x \wedge (x \to y)) \to z =$ B5) $x \to ((x \to y) \to z) =$ B9)$(x \to y) \to (x \to z)$ (4).

Furthermore $(x \to y) \to (x \to z) =$ B5) $(x \wedge (x \to y)) \to z =$ B3) $(x \wedge (\sim x \vee y)) \to z =$ B5) $(\sim x \vee y) \to (x \to z)$ (6)

Since $y \leq \sim x \vee y$ from B13 we have $(\sim x \vee y) \to z \leq y \to (x \to z)$ (7). Finally, from (7), B9 and (6) $(x \to y) \to (x \to z) \leq x \to (y \to z)$ (8). B19 is a consequence of (40 and (8).

In 2.4. we give an equivalent definition of N-lattices.

2.4.Theorem: In any G.N-L. the condition N6 of 1.1. is equivalent to

B20') $(x \to z) \wedge (y \to z) \leq (x \vee y) \to z$

Proof. In [13] A.Monteiro proved that B20' is a consequence of N6. For the converse, suppose B20', then.

$1 =$(B1,B14) $(\sim x \to (x \to y)) \vee (\sim x \to (x \to \sim y)) =$ B5) $((\sim x \wedge x) \to y) \vee ((\sim x \wedge x) \to y) \leq$ B20) $(\sim x \wedge x) \to (y \vee \sim y)$ (1).

Furthermore $\sim ((x \wedge \sim x) \wedge (y \vee \sim y)) \to \sim (x \wedge \sim x) =$ (N5,N4) $(M(\sim y \wedge y) \vee (\sim x \;5\; x)) \to (\sim x \vee x) =$ (B20,B20')

$((\sim y \wedge y) \to (\sim x \vee x)) \wedge ((\sim x \vee x) \to (\sim x \vee x)) =$ ((1),B1)1 (2). From (1),(2) and B19, $\sim x \wedge x \leq y \vee \sim y$.

3. Deductively semisimple G.N.-L. and modal tetravalued algebras

3.1. Definition: A G.N.-L. is said to be deductively semisimple if the condition

S) $(x \to y) \to x = x$

is verified

3.2. Remark: As $\rightarrow$ verifies B6, B8 and B19, we can use the results obtained in [14], p.427-431 (See also [17],p.23-31). In particular, if A is a deductively semisimple G.N-L. then:

1) A proper deductive system M of A is maximal if and only if $x \in A\text{-}M$ and $y \in A$ imply $x \rightarrow y \in M$.

2) Every proper deductive system of A is an intersection of maximal deductive systems.

3) If $x \in A$, $x \neq 1$, there exists a maximal deductive system M of A such that $x \notin M$.

3.3.Lemma: If A is a deductively semisimple G.N-L., then for $x,y \in A$ the following condition is verified: S1) $(x \rightarrow y) \vee x = 1$.

Proof. Consider the element $\beta = (x \rightarrow y) \vee x$. If we suppose $\beta \neq 1$, by 3.2.3 º) there would exist a maximal deductive system M such that $\beta \notin M$ (1). Then, $x \rightarrow y \notin M$ (2), and $x \notin M$ (3). From (3) and 3.2.1 º) we would have $x \rightarrow y \in M$ (4). (2) and (4) contradict each other.

3.4. Theorem: If $(A,1,\sim, \wedge, \vee, \nabla)$ is a modal tetravalued algebra, and $\rightarrow$ is the operation defined by the formula (I), the algebra $(A,1,\sim, \wedge, \vee, \rightarrow)$ is a deductively semisimple G.N-L., verifying (P) $\nabla x = \sim x \rightarrow 0$.

Proof. Let us prove that the formulas B2,B3,B4,B5 and S hold:

B2) By M3, $\sim x \leq \nabla \sim x$ (1), from (1) and I, $\sim x \vee y \leq x \rightarrow y$

B3) $x \wedge (x \rightarrow y) =$ (I) $x \wedge (\nabla \sim x \vee y) = (x \wedge \nabla \sim x) \vee (x \wedge y) =$ M11) $(\sim x \wedge x) \vee (x \wedge y) = x \wedge (\sim x \vee y)$

B4) $x \rightarrow (y \wedge z) =$ (I) $\nabla \sim x \vee (y \wedge z) = (\nabla \sim x \vee z) \wedge (\nabla \sim x \vee y) =$ (I) $(x \rightarrow y) \wedge (x \rightarrow z)$

B5) $x \rightarrow (y \rightarrow z) =$ (I) $\nabla \sim x \vee (\nabla \sim y \vee z) =$ M7) $\nabla (\sim x \vee \sim y) \vee z =$ (N5, I) $(x \wedge y) \rightarrow z$

S) From I $x \leq (x \rightarrow y) \rightarrow x$. On the other hand $(x \rightarrow y) \rightarrow x =$ I) $\nabla \sim (\nabla \sim x \vee y) \vee x = \nabla (\sim \nabla \sim x \wedge \sim y) \vee x \leq$ (M9, M10) $(\sim \nabla \sim x \wedge \nabla \sim y) \vee x \leq \sim \nabla \sim x \vee x =$ M4) x

The proof of P is straightforward.

3.5. Theorem: If $(A,1, \sim, \wedge, \vee, \rightarrow)$ is a deductively semisimple G.N-L. and ∇ is the operation defined by the formula (P), the system $(A,1, \sim, \wedge, \vee, \nabla)$ is a moda l tetravalued algebra verifying $x \rightarrow y = \nabla \sim x \vee y$.

Proof. Let us prove the formula M1, M2 and I:

M1) $\sim x \vee \nabla x = 1$

$\sim x \vee \nabla x =$ (P) $\sim x \vee (\sim x \rightarrow 0) =$ S1)1

M2) $\sim x \wedge \nabla x = x \wedge \sim x$

$\sim x \wedge \nabla x =$ (P) $\sim x \wedge (\sim x \rightarrow 0) =$ B3) $\sim x \wedge (\sim\sim x \vee 0) =$N4) $\sim x \wedge x = x \wedge \sim x$

I) $x \rightarrow y = \nabla \sim x \vee y$

By S1, $x \vee (x \rightarrow y) = 1$ (1). From (P) and S1 $x \vee \nabla \sim x = 1$ (2). From (1), (2) and N1 $x \vee (x \rightarrow y) = x \vee (\nabla \sim x \vee y)$ (a). On the other hand $x \wedge (\nabla \sim x \vee y) = (x \wedge \nabla \sim x) \vee (x \wedge y) =$ M11) $(x \wedge \nabla x) \vee (x \wedge y) = x \wedge (\nabla x \vee y) =$ B3) $x \wedge (x \rightarrow y)$ (b). From (a),(b) we have I.

3.6. Remark: It is well known that in N-lattices, congruences are determined by deductive systems, where $\equiv_D$ is a congruence if and only if there exists a deductive system D such that $x \equiv_D y$ if and only if $x \rightarrow y \in D$ and $y \rightarrow x \in D$, $\sim x \rightarrow \sim y \in D$, $\sim y \rightarrow \sim x \in D$. This result does not hold in G.N-L. as can be easily accomplished in example 2.2..

In the case of deductively semisimple G.N-L. the following result can be 3.6.1. If A is a G.N-L., $\equiv D$ is a congruence on A if and only if there exists a deductive system D such that $x \equiv D\ y$ if and only if $x \rightarrow y \in D$, $y \rightarrow x \in D$, $\sim(x \wedge y) \rightarrow \sim x \in D$, $\sim(x \wedge y) \rightarrow \sim y \in D$.

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