Revista de la
Unión Matemática Argentina
Vol. 35 1990.

PARTIAL ORDER AND OPTIMAL CONTROL

EMILIO O. ROXIN

Abstract. In optimal control, a given process is considered and the goal is to control this process in such a way that certain function (the "cost" function) is optimized at the end-time. For some problems the process is assumed to go on forever; these are called "infinite horizon problems" and in that case, the measure of success depends on the asymptotic behavoir of the cost function. This leads to an ordering of preference for selecting controls, which can be defined in different ways; such possible orderings may lead to a partial order. In this paper partial order relations are studied in the context of infinite horizon control problems.

I. Sets with a partial order

Given a set $S = \{x, y, z, \ldots\}$, a partial order (p.o.) on it is a relation denoted by $<$, defined for some pairs $x, y \in S$, such that:

a) $x < x$ for every x (reflexive);
b) $x < y$ and $y < x$ imply $x = y$ (antisymmetric),
c) $x < y$ and $y < z$ imply $x < z$ (transitive).

Consider, for example, the set of functions of a real variable, defined in the interval $[0,\infty)$, on which additional constraints may be imposed (for example to be continuous, or L-integrable etc.). The simplest way to define a p.o. on this set is:

$$x < y \quad \text{if} \quad x(t) \le y(t) \quad \text{for every} \quad t \in [0, \infty). \tag{1}$$

In the case of L-integrable functions, one should consider this last inequality as a.e.

It is easy to check that this definition satisfies the above conditions for a p.o.

A partially ordered set is a "lattice" if for any two elements x, y of S there is a least upper bound (l.u.b.) and a greatest lower bound (g.l.b.), i.e. two elements u, v such that $v < x < u$ and $v < y < u$, and for any u', v' with the same properties, $v' < v$, $u < u'$.

The above mentioned functions, with the p.o. given by (1), constitute a "lattice". Indeed, while not every two functions x, y are related in this p.o., there is a least upper bound and a greatest lower bound for x and y, given by

$$\text{l.u.b.}(x, y) = u, \text{ where } u(t) = \max(x(t), y(t)),$$
$$\text{g.l.b.}(x, y) = v, \text{ where } v(t) = \min(x(t), y(t)).$$

If the functions of S are assumed continuous, then these l.u.b. and g.l.b. will also be continuous. If the functions are assumed to be only L-integrable, then the same will be true, considering the definitions of u and v as defined a.e. If, on the other hand, the functions of S are required to be differentiable, than the l.u.b. and g.l.b. will not be elements of S, which then will not be a lattice.

An element a of a partially ordered set S is greatest (smallest), if for any $x \in S$, $x < a$ ($x > a$).

An element a of a partially ordered set S is maximal (minimal), if $x > a$ ($x < a$) implies $x = a$.

As standard references for the definitions of partial order, the reader is referred to [1], [2], [3].

II. Asymptotic equivalence

Assume that the elements of S are the absolutely continuous functions of a real variable defined on $[0,\infty)$. Assume that we are interested in ordering them according to their behavoir for large t. A suitable definition will then be:

(2) $$x < y \text{ if there exists a } T \text{ such that for all } t \geq T,\ x(t) \geq y(t).$$

This definition satisfies conditions (a) and (c) above, but not (b). It defines, therefore, a "quasi-ordering" [3]. To get a p.o. we should consider the set S of equivalence classes on S, where

(3) $$x \sim y \text{ if } x < y \text{ and } y < x.$$

Or, equivalenty, $x \sim y$ if $\exists\ T$ such that for all $t \geq T$, $x(t) = y(t)$.

Two functions $x(t)$ and $y(t)$ which satisfy (3) will be defined to be "infinite time equivalent".

It is interesting to notice that it is possible that a pointwise convergent sequence $x_n \to \bar{x}$ has all its members x_n belonging to the same equivalence class, but the limit $\bar{x}$ does not belong to the same class. A counterexample is :

$$\begin{aligned} x_n(t) &= 0 && \text{for } 0 \leq t \leq n, \\ &= t - n && \text{for } n \leq t \leq n+1, \\ &= 1 && \text{for } t \geq n+1. \end{aligned}$$

Obviously, $\bar{x}(t) = 0$ for all t, while the equivalence class is characterized by $x(t) = 1$ for large t.

In this quasi-ordering, the equivalence classes could be described as the "tails" of the functions in the class (their behavoir in $M \leq t \leq \infty$, with $M \to \infty$).

Similarly, in the same class of functions, a quasi-order can be defined by

(4) $$x < y \text{ if there is an } \varepsilon > 0 \text{ such that for } 0 \leq t \leq \varepsilon,\ x(t) \leq y(t).$$

In this case the equivalence classes are defined by

$$x \sim y \text{ if } \exists\ \varepsilon > 0 \text{ such that for all } t \in [0,\varepsilon],\ x(t) = y(t).$$

These equivalence classes can be described as the "germs" of the functions at $t = 0$.

A somewhat different p .o . is obtained by defining

(5) $x < y$ if, given $\varepsilon > 0$, $\exists\, T$ such that $t > T$ implies $x(t) \leq y(t) + \varepsilon$.

Equivalently, $x < y$ if $\limsup\ (x(t) - y(t)) \leq 0$ for $n \to \infty$.

The corresponding equivalence relation is given by

$$x \sim y \text{ if } \lim\ [x(t) - y(t)] = 0 \quad \text{for } n \to \infty .$$

Two such functions $x \sim y$ will be said to be asymptotically equivalent.
Coming back to the p .o . defined by (2) above, consider two functions x, y which are not comparable (with respect to that p .o .) . In this case, given any number T, there is some $t \geq T$ such that $x(t) > y(t)$. Hence it is possible to construct a sequence $t_n \to \infty$ such that for all n, $x(t_n) > y(t_n)$. Similarly, there is another sequence $t_n' \to \infty$ such that $x(t_n') < y(t_n')$. If the objective is to minimze $x(t)$ at the end time of the process, then either x or y is preferable, depending on the timet at which the process ends.

Infinite horizon optimal control problems

The simplest infinite horizon optimal control problem is given by a differential equation

(6) $\dot{x} = f(x,u)$,

representing the dynamics of the system under consideration. Here $x = x(t)$ in an n-vector, the "state", t is the time and $u = u(t)$ is an m-vector, the "control" function. It is assumed that the function $f(x,u)$ is given, and also the initial state

(7) $x(0) = x_0$.

The control function $u(t)$ is expected to be chosen from a family of admissible controls. The class of admissible controls is usually defined by

(8) $u(t)$ measurable with values in $x(t) \in U$.

where the set $U \subset R^m$ is assumed compact.
There is a "cost functional"

(9) $J_u(T) = \int_0^T f_0\,(x(t)\,,u(t))\,dt$

associated with the process, which should be minimized in an appropriate sense.
In (9), f_0 is a given scalar-valued function.
The functions $f(x,u)$, $f_0\,(x,u)$, being given, may be assumed to be as "nice" as needed. Usually f and f_0 are assumed to be C^1, which insures that for every admissible control $u(t)$ there exists a unique solution $x(t)$ of (6), (7). It is also necessary to assume a growth condition on $f(x,u)$, for example that f does not grow faster than linear in x, in order to assure that solutions $x(t)$ do not "escape to infinity in finite time".
For the control $u(t)$, strongly restrictive conditions may not be imposed; indeed, for many simple and interesting problems the optimal control turns out to be discontinuous.

Definitions of optimally

Associated with each admissible control $u(t)$ there is a "trajectory" $x(t)$, solution of (6), (7), and a cost function $J(T)$ given by (9). Here, the pairs $(x,u) = (x(t), u(t))$, $0 \geq t$, will be ordered according to the desirability of their cost function $J_u(T)$ (9).

The following definitions of optimality appear in the literature ([4], [5]):

The pair $(x^*(t), u^*(t))$, $0 \leq t$, is called

O) Overtaking optimal if, for any admissible pair (x, u), there is a T_u such that for every $T \geq T_u$, $J_u(T) \geq J_{u*}(T)$.

UO) Uniformly overtaking optimal, if there is some T_0 such that for every $T \geq T_0$ and every admissible pair (x, u), $J_u(T) \geq J_{u*}(T)$.

C) Catching up optimal if, for any admissible pair (x, u) and every $\varepsilon > 0$, there is a $T_{u,\varepsilon}$, such that for every $T \geq T_{u,\varepsilon}$, $J_u(T) \geq J_{u*}(T) - \varepsilon$.

UC) Uniformly catching up optimal, if for every $\varepsilon > 0$ there is a T_ε such that for every admissible pair (x, u) and for every $T \geq T_\varepsilon$, $J_u(T) \geq J_{u*}(T) - \varepsilon$.

SO) Sporadically overtaking optimal, if for every T_0 and any admissible pair (x,u), there is a $T_u \geq T_0$ such that $J_u(T_u) \geq J_{u*}(T_u)$.

USO) Uniformly sporadically overtaking optimal, if for every T_0 there is some $T \geq T_0$ such that for every admissible pair (x, u), $J_u(T) \geq J_{u*}(T)$.

SC) Sporadically catching up optimal, if for every T_0, every $\varepsilon > 0$ and every admissible pair (x, u), there exists a $T_{u,\varepsilon} \geq T_0$ such that $J_u(T_{u,\varepsilon}) \geq J_{u*}(T_{u,\varepsilon}) - \varepsilon$.

USC) Uniformly sporadically catching up optimal, if for every T_0 and every $\varepsilon > 0$, there is a $T_\varepsilon \geq T_0$ such that for every admissible pair (x, u) $J_u(T_\varepsilon) \geq J_{u*}(T_\varepsilon) - \varepsilon$.

An example

Consider the damped oscillator with forcing control

$$\begin{aligned} \dot{x}_1 &= -k\,x_1 + \omega\,x_2 \\ \dot{x}_2 &= -\omega\,x_1 - k\,x_2 + u, \end{aligned} \tag{10}$$

where $|u| \leq 1$ is the control. Take

$$x(0) = 0 \tag{11}$$

and

$$J_u(T) = -x_1(T) \to \min \tag{12}$$

(hence there is no need to consider $f_0(x, u)$ as in (9)).

It is elementary to see that for $u = 0$, the solutions of (10) are the spirals

$$x_1 = r_0 e^{-kt} \sin(\omega t - \theta), \quad x_2 = r_0 e^{-kt} \cos(wt - \theta) \tag{13}$$

which tend asymptotically to the focus (0,0).

The cases $u = 1$ and $u = -1$ (constant, bang-bang control) can be reduced to the $u=0$ case by the substitutions

$$x_1 = \bar{x}_1 \pm \omega / (k^2 + \omega^2), \qquad x_2 = \bar{x}_2 \pm k / (k^2 + \omega^2),$$

hence the corresponding solutions are again spirals converging to the foci

$$(x_{10}, x_{20}) = \pm (\omega/(k^2 + \omega^2), k/ (k^2 + \omega^2)).$$

According to the maximum principle of Pontryagin, any optimal control must be bang-bang, switching from $u = 1$ to $u = -1$ and back, with time intervals π/ω between any two switches. With such a control, the corresponding trajectory tends asymptotically to a closed curve constituted by two arcs of spiral of the families described above (see figures 1 and 2).

A solution will be optimal for the (finite) end-time T, if the switching times are adjusted in such a way that $x_1(T)$ is a maximum. This will happen at switching points. It is easy to see that any two such bang-bang solutions will periodically catch up with each other, hence neither of them will be optimal for all sufficiently large end-times T. In other words, this solution will not be overtaking optimal (definition O) nor catching up optimal (definition C). Hence there is no "greatest" (nor "smallest") solution in the corresponding p.o. defined by (2), but such solutions are not comparable. Hence each of them is sporadically catching up optimal and therefore maximal (or minimal) in the p.o.

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Figure 1 : limit cycle in x_1 - x_2 plane
Figure 2 : plot of $x_1(t)$ with optimal $x_1(T)$

University of Rhode Island
Kingston, RI, U.S.A.

Recibido por U.M.A. el 16 de Mayo de 1989.

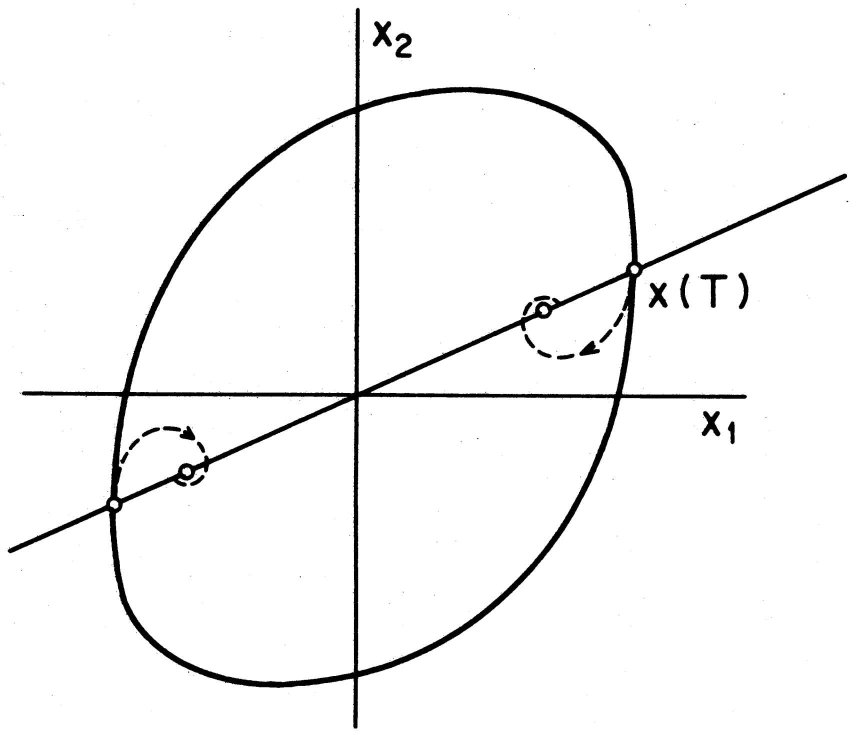

Fig.1. Limit cycle in the x_1 - x_2 plane.

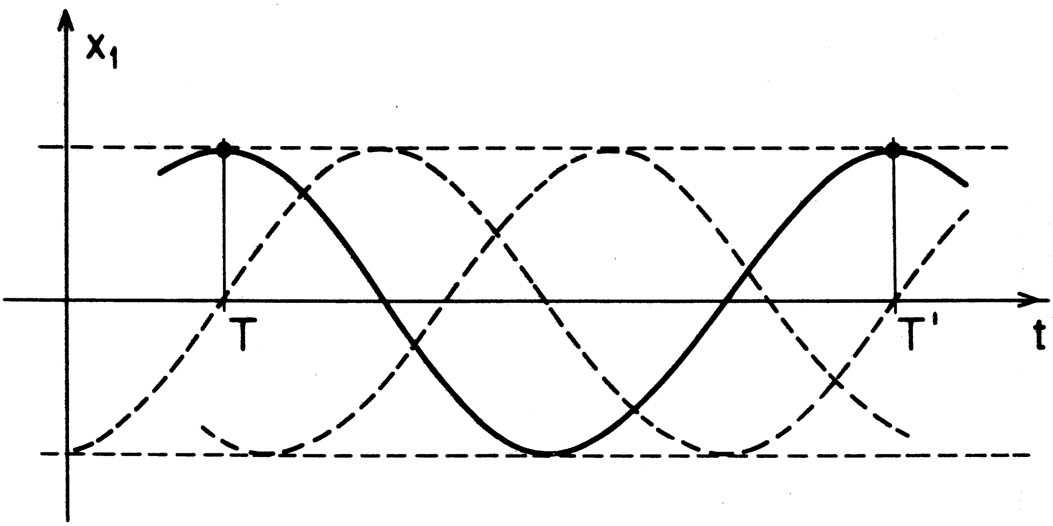

Fig. 2. Plot of $x_1(t)$ with optimal $x_1(T)$