Revista de la
Unión Matemática Argentina
Vol. 35 1990.

A COMBINATORIAL APPROACH TO THE GENERALIZED SYLVESTER'S PROBLEM

FAUSTO A. TORANZOS

If A *is a finite set of points in the plane, no two of them on a vertical line, and card* A $\geq n+1$, *then there exists a polynomial of degree less than* $n+1$ *such that its graph includes exactly* $n+1$ *points of* A, *unless all the points of* A *lie on the graph of such a polynomial.*

The case $n = 1$ of this statement is the classical Sylvester's Problem stated in 1893 and solved by T. Gallai in 1933. Peter Borwein has recently solved the generalized form of this problem in [1]. His proof relies on the metrical structure of the plane and the theory of uni-modal Haar spaces of continuous functions. We intend to show that this is a purely combinatorial problem, by proving the initial statement of this note without the use of metrical or topological tools.

If $\mathcal{P}_n$ denotes the real vector space of all the polynomials of a real variable having degree less than or equal to n, we can define a mapping (actually a linear isomorphism) $\Psi : \mathcal{P}_n \to \mathbf{R}^{n+1}$ such that if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

then

$$\Psi(f) = (a_0 ;\ a_1 ;\ a_2 ;\ \ldots ;\ a_n).$$

We also define $\lambda : \mathbf{R} \to \mathbf{R}^{n+1}$ such that $\lambda(t) = (1 ;\ t ;\ t^2 ; \ldots ; t^n)$.

If P is a point of the plane, let H_P denote the set of all elements of $\mathcal{P}_n$ whose graphs pass through P.

Lemma 1. *For every point* P *of the plane,* H_P *is a hyperplane of* $\mathcal{P}_n$.

Proof: If $f \in \mathcal{P}_n$ and $t \in \mathbf{R}$, the specialization of f at t is

$$f(t) = \langle \Psi(f) \mid \lambda(t) \rangle$$

where $\langle \mid \rangle$ denotes the inner product in $\mathbf{R}^{n+1}$. Let $P = (x_0 ;\ y_0) \in \mathbf{R}^2$, then

$$H_P = \{ f \in \mathcal{P}_n \mid f(x_0) = y_0 \} = \{ f \in \mathcal{P}_n \mid \langle \Psi(f) \mid \lambda(x_0) \rangle = y_0 \}$$

Clearly this is the equation of a hyperplane.

In the sequel, when a point of the plane is described using a subscript (P_j for example), the corresponding hyperplane in $\mathcal{P}_n$ that Lemma 1 provides will be denoted using the same subscript (H_j in our example). Two previously known results will be quoted here as lemmas for future reference.

Lemma 2. *Given* $n+1$ *points of the plane* $\{P_0; P_1; \dots; P_n\}$, *no two of them on a vertical line, the corresponding hyperplanes* $H_0, H_1, \dots, H_n$ *meet precisely in one point of* $\mathcal{P}_n$.

Proof: This is the well known Lagrange's interpolation theorem.

Lemma 3. *Let* $\mathcal{S}$ *be a finite family of hyperplanes in a finite-dimensional real projective space, and let* $\mathcal{M}$ *be the set of points determined as intersections of the members of* $\mathcal{S}$. *Assume that* $\mathcal{M}$ *contains more than one point. Then there exists* $Q_0 \in \mathcal{M}$ *such that every member of* $\mathcal{S}$ *that includes* Q_0, *except precisely one of them, contains a certain line of the space.*

Proof: This statement is the dual form of the main theorem of [2]. It is noteworthy that this result has been obtained in a purely combinatorial context, without any topological requirements whatsoever.

Theorem. *Let* A *be a finite set of at least* $n+1$ *points of the plane, no two of them on the same vertical line. Then there exists a real polynomial having degree less than or equal to n whose graph includes precisely* $n+1$ *points of* A, *unless all the points of* A *lie on the graph of such a polynomial.*

Proof: Let $A = \{P_i \mid i \in I\}$ and define $\mathcal{S} = \{H_i \in \mathcal{P}_n \mid i \in I\}$. In order to apply the previous lemma we consider the space $\mathcal{P}_n$ embedded in the real $(n+1)$ - dimensional projective space. As in Lemma 3 call $\mathcal{M}$ the set of points of the projective space determined as intersections of subfamilies of $\mathcal{S}$, and assume that $\mathcal{M}$ has more than one point. We claim that all the members of $\mathcal{M}$ are proper points of the projective space. Otherwise suppose there would exist a subfamily $\mathcal{F} \subset \mathcal{S}$ such that the intersection of its members is a point of the improper hyperplane of the projective space. Considering the members of $\mathcal{F}$ as subsets of the vector space $\mathcal{P}_n$ their intersection would be empty, thus leading easily to a contradiction with Lemma 2. Hence we have proved that $\mathcal{M} \subset \mathcal{P}_n$. Let $Q_0 \in \mathcal{M}$ be the exceptional point mentioned in the thesis of Lemma 3. Denote $\mathcal{S}_0$ the family of all the hyperplanes of $\mathcal{S}$ that includes Q_0, and $\mathcal{F}_0$ the subfamily obtained deleting precisely one of the hyperplanes and such that the intersection of its members include a line L. In order to avoid a contradiction with Lemma 2, F_0 must have less than $n+1$ members, whence card $(\mathcal{S}_0) = n+1$. It is easy to see that

$$f = \Psi^{-1}(Q_0)$$

verifies the thesis. The exceptional case where $\mathcal{M}$ is reduced to a single point is taken care by the last part of the statement of this theorem.

References.

[1] BORWEIN, P.B. *On Sylvester's Problem and Haar spaces,* Pacific J. of Math. 109 (1983), 275 - 278.

[2] HANSEN, S. *A generalization of a theorem of Sylvester on the lines determined by a finite point set*, Math. Scand. 16 (1965), 175 - 180.

Departamento de Matemática
Universidad de Buenos Aires
Pabellón 1, Ciudad Universitaria
(1428) Buenos Aires. Argentina.

Recibido por U.M.A. en mayo de 1989.