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COHOMOLOGICAL TRIVIALITY BY SPECTRAL METHODS, II

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To the respected memory of Julio Rey Pastor

Abstract. The spectral sequences associated to a group extension are used to give a proof (different from a previous one) of the twins' criterion for cohomological triviality of modules over a finite group, stated in its strong form. A discussion of the conjecture extending the criterion is included.

The criterion. As usual, the Tate cohomology will be denoted by $\hat{H}$, while H, with upper or lower indices, will be used to represent the ordinary cohomology and homology, respectively.

The twins' criterion is, of course, the following.

Theorem (Nakayama-Tate). *Let* G *be a finite group and* A *a* G*-module. If, for each prime* p, *there exists an integer* r_p *(depending on* p*) such that*

$$\hat{H}^{r_p}(S_p, A) = \hat{H}^{r_p+1}(S_p, A) = 0,$$

where S_p *is a Sylow* p*-subgroup of* G, *then* A *is cohomologically trivial.*

This was proved in [6] by using the cohomology version of the Hochschild-Serre spectral sequence. The proof offered here also rests heavily on spectral techniques. Although more complicated than that in [6], it provides some support to the idea of establishing the Tannaka's conjecture by such techniques.(This will be explained later).

The proof. Borrowing from [6], it suffices to consider the case where G is a p-group, for some prime p, and to establish the following two statements, for some integer r :

(i) *If* $\hat{H}^r(G, A) = \hat{H}^{r+1}(G, A) = 0$, *then* $\hat{H}^r(S, A) = \hat{H}^{r+1}(S, A) = 0$ *for all subgroups* S *of* G.

(ii) *If* $\hat{H}^r(G, A) = \hat{H}^{r+1}(G, A) = 0$, *then* $\hat{H}^n(G, A) = 0$ *for all integers* n.

In turn, (i) *reduces to show:*

(i_c) *If* $H^1(G, A) = H^2(G, A) = 0$, *then* $H^2(S, A) = 0$.

(i_h) *If* $H_1(G, A) = H_2(G, A) = 0$, *then* $H_2(S, A) = 0$.

This reduction follows easily by considering dimension shifters B and C such that

$$\hat{H}^n(S, A) \approx \hat{H}^{n+1-r}(S,B) \text{ and } \hat{H}^n(S, A) \approx \hat{H}^{n-3-r}(S,C).$$

Proof of (i_c). The periodicity of the cohomology of finite cyclic groups will be applied to the initial term of the cohomology spectral sequence of a group extension by such a factor.

Consider, for the moment, the situation of G any group and S a normal subgroup of G such that G/S is a finite cyclic group. Let (E (A) , H (A)) be the Hochschild-Serre spectral sequence in cohomology associated to a given G-module A, so that

$$E_2^{p,q}(A) = H^p(G/S, H^q(S, A)) \text{ and } H^n(A) = H^n(G, A).$$

The choice of a generator of G/S determines a cohomology class c in $H^2(G/S, \mathbf{Z})$ yielding the periodicity map

$$H^p(G/S, H^q(S, A)) \to H^{p+2}(G/S, H^q(S, A)), \quad x \to x \vee c,$$

where

$$\vee : H^p(G/S, H^q(S, A)) \times H^{p'}(G/S, \mathbf{Z}) \to H^{p+p'}(G/S, H^q(S, A))$$

is the cup product for the usual G/S-pairing $H^q(S, A) \times \mathbf{Z} \to H^q(S, A)$ (see [9, Remark, p. 141] and [3, Exercise 6, p. 263]). The map just defined is an isomorphism for $p > 0$; but it only is an epimorphism for $p = 0$. Concerning $\vee$, this product can also be obtained as follows. The usual pairing $A \times \mathbf{Z} \to A$ provides the cup product $H^q(S, A) \times H^{q'}(S, \mathbf{Z}) \to H^{q+q'}(S, A)$, which regarded as a G/S-pairing induces the product

$$\cup : H^p(G/S, H^q(S, A)) \times H^{p'}(G/S, H^{q'}(S, \mathbf{Z})) \to H^{p+p'}(G/S, H^{q+q'}(S, A)).$$

Since the pairing defining $\cup$ yields, for $q'=0$, that defining $\vee$, from the uniqueness of cup products [5, Theorem 2,p .104] it follows that $\cup$, with $q'= 0$, gives $\vee$. Moreover, $\cup$ is related to the product structure of the spectral sequence: if the map

$$\cdot : \quad E_2^{p,q}(A) \times E_2^{p',q'}(\mathbf{Z}) \to E_2^{p+p',q+q'}(A)$$

is the product of the initial terms, then $x \cdot y = (-1)^{pq'} x \cup y$ (see[4, remark following Theorem 3,p .126] or [5,Theorem 5, p . 162] for the full statement). The preceding considerations allow to describe the periodicity map as the morphism

$$E_2^{p,q}(A) \to E_2^{p+2,q}(A), \quad x \to x.c.$$

Finally, the differentiation rule for products shows the commutativity of the diagram

$$\begin{array}{ccc} E_2^{p,q}(A) & \to & E_2^{p+2,q-1}(A) \\ \downarrow & & \downarrow \\ E_2^{p+2,q}(A) & \to & E_2^{p+4,q-1}(A), \end{array}$$

where the horizontal arrows represent the differentials.

Going to the proof itself, G is assumed again to be a p-group, and S a (normal) subgroup of index p in G (this additional reduction was also used in [6]). Since integer coefficients will not appear explicitly, the letter A will be supressed in the notation of the spectral sequence.

In the commutative diagram·

$$\begin{array}{ccc} E_2^{0,1} & \rightarrow & E_2^{2,0} \\ \downarrow & & \downarrow \\ E_2^{2,1} & \rightarrow & E_2^{4,0} \end{array}$$

the left-hand arrow is surjective and the right-hand arrow is injective (actually, an isomorphism). Furthermore, the differential $d_2^{0,1}$ is a monomorphism, since $E_3^{0,1} = 0$ under the assumption that $H^1 = 0$. The conclusion is that $d_2^{2,1}$ is also a monomorphism, whence $d_2^{0,2} = 0$. Therefore, $E_3^{0,2} = E_2^{0,2}$.

On the other hand, $E_4^{0,2} = E_3^{0,2}$, because $E_2^{1,0} = 0$ and hence, $E_3^{3,0} = 0$.

Taking into account that $E_4^{0,2} = 0$, since $H^2 = 0$, it is proved that $E_2^{0,2} = 0$. This suffices to conclude that $H^2(S,A) = 0$ (see [6] for the details).

Proof of (i_h). It can be arranged in perfect duality with the proof of (i_c), and will be included for the convenience of the reader, which is refered to [2] for a direct treatment of the cap product.

In the situation at the beginning of the previous proof, consider the morphism

$$H_p(G/S, H_q(S, A)) \rightarrow H_{p-2}(G/S, H_q(S, A)), \; x \rightarrow x \wedge c,$$

where

$$\wedge : H_p(G/S, H_q(S, A)) \times H^i(G/S, \mathbf{Z}) \rightarrow H_{p-i}(G/S, H_q(S, A))$$

is the cap product induced by the usual G/S - pairing $H_q0(S, A) \times \mathbf{Z} \rightarrow H_q(S, A)$. For $p > 2$, this morphism coincides, up to sign, with the periodicity map obtained from the Tate product; in particular, it is bijective [3, Exercise 6, p. 263 (the reader is warned to save the typographical errors concerning cap product)]. Furthermore, it is injective for p=2. For uniqueness reasons (or direct verification), $\wedge$ also comes from the cap product

$$\cap : H_p(G/S, H_q(S, A)) \times H^i(G/S, H^j(S, \mathbf{Z})) \rightarrow H_{p-i}(G/S, H_{q-j}(S, A))$$

corresponding to the G/S-pairing $H_q(S, A) \times H^j(S, \mathbf{Z}) \rightarrow H_{q-j}(S, A)$ induced by the G-pairing $A \times Z \rightarrow A$ [2, Lemma 1.2.6, p. 378].

Now, let (E,H) be the Hochschild-Serre homology spectral sequence of the G-module A, so that

$$E^2_{p,q} = H_p(G/S, H_q(S, A)) \text{ and } H_n = H_n(G,A).$$

If

$$\cdot : E^2_{p,q} \times E_2^{i,j}(\mathbf{Z}) \rightarrow E^2_{p-i,\,q-j}$$

is the cap product of the initial terms [2, §1.3, pp. 378-380], then $x \cdot y = (-1)^{qi}\, x \cap y$ [2, Lemma 1.3.2, p .380]. Therefore, the morphism defined above coincides with

$$E^2_{p,q} \to E^2_{p-2,q},\quad x \to x \cdot c.$$

The diagram

$$\begin{array}{ccc} E^2_{p,q} & \to & E^2_{p-2,q+1} \\ \downarrow & & \downarrow \\ E^2_{p-2,q} & \to & E^2_{p-4,q+1} \end{array}$$

is commutative, according to the differentiation formula

$d^r_{p-i,q-j}(x \cdot y) = (-1)^{i+j}\, d^r_{p,q}\, x \cdot y + x \cdot d^{i,j}_r\, y$ (cf. [2, § 1.3, (*), p. 379]).

In the setting of (i_h), with the previous reductions in force, a perfect duality works as follows: In the commutative diagram

$$\begin{array}{ccc} E^2_{4,0} & \to & E^2_{2,1} \\ \downarrow & & \downarrow \\ E^2_{2,0} & \to & E^2_{0,1}\ , \end{array}$$

the left-hand arrow is surjective (actually, an isomorphism) and the right-hand arrow is injective. Furthemore, the differential $d^2_{2,0}$ is an epimorphism, since $E^3_{0,1} = 0$ under the assumption that $H_1 = 0$. The conclusion is that $d^2_{4,0}$ is also an epimorphism, whence $d^2_{2,1}$. Therefore, $E^3_{0,2} = E^2_{0,2}$.

On the other hand, $E^4_{0,2} = E^3_{0,2}$, because $E^2_{1,0} = 0$ and hence, $E^3_{3,0} = 0$.

Taking into account that $E^4_{0,2} = 0$, since $H_2 = 0$, it is proved that $E^2_{0,2} = 0$. This suffices to conclude that $H_2\,(S,A) = 0$ (see [6] for the details).

Of course, the duality was partially masked by the notation of the differentials.

Proof of (ii). By dimension shifting, it can be assumed that $r > 0$, for instance. Arguing by induction on the exponent of the order of G, suppose that G is nontrivial and take a subgroup S of index p. By (i) and the inductive assumption, $\hat{H}^n$ (S, A) = 0 for all integers n. Thus, the corresponding cohomology spectral sequence (E , H) collapses (i.e., $E_2^{p,q} = 0$ for $q > 0$) and provides the isomorphisms $E_2^{n,0} \approx H^n$. Since $H^r = H^{r+1} = 0$, it is clear that $E_2^{r,0} = E_2^{r+1,0} = 0$. Hence, $E_2^{n,0} = 0$ for $n > 0$ (periodicity), that is, $\hat{H}^n\,(G, A) = 0$ for $n > 0$.

Take now a dimension shifter B satisfying $\hat{H}^n\,(G, B) \approx \hat{H}^{n+t}\,(G, A)$ for some integer $t > r + 2$. Thus, $H_s\,(G, B) = H_{s+1}\,(G, B) = 0$ for some integer $s > 0$. A dual argument using the homology spectral sequence shows that $\hat{H}^n\,(G, B) = 0$ for $n < -1$. Therefore, $\hat{H}^n\,(G,A) = 0$ for $n \leq 0$.

Note that the proof presents (ii) as a formal consequence of (i).

Tannaka's conjecture . A natural conjecture arises from the criterion by replacing in its statement the number 1 by an odd integer i_p (supposed positive for convenience). Of course, even integers are excluded for periodicity reasons.

According to Uchida [11], this conjecture is due to Tannaka, who established affirmatively a weak form: fixed dimensions and all subgroups in the hypothesis. This weak form was also obtained by Onishi [7]. Furthermore, Uchida [11] proved two special cases of the conjecture:

(1) A is finitely generetated and $i_p \leq 3$ for every prime p.

(2) The Sylow subgroups of G are direct products of two cyclic groups.

The arrangement of the present proof of the criterion can easily be adapted to the conjecture by replacing the number 1 by a positive odd integer i in the statements of (i) and (ii). In that situation, (ii) still follows from (i). Therefore, it suffices to prove:

$$\text{If} \quad \hat{H}^r(G, A) = \hat{H}^{r+i}(G, A) = 0, \quad \text{then } \hat{H}^r(S, A) = \hat{H}^{r+i}(S, A) = 0,$$

provided that G is a p-group and S is a subgroup of index p in G.

The author believes that reductions like (i_c) and (i_h) might be established by using periodicity arguments in the spectral sequences of a group extension by a finite cyclic factor.

The study of these periodicity questions is not new. The cohomology case, for p-groups and integers mod p as coefficients, was first considered by Serre (see [10, Remark, p . 418]), and then, by Quillen-Venkov [8] and Alperin-Evens [1]. Similar results for integer coefficients were exposed by the author in an algebra seminar held at Universidad de Buenos Aires in 1971.

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