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INVARIANT POLYNOMIALS FOR A MAXIMAL UNIPOTENT SUBGROUP OF SL(n,C)

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Abstract. Let N be a maximal unipotent subgroup of SL(n, C).
Inside the ring of N-invariant polynomials $P[\mathfrak{sl}(n, C)]^N$ we consider the subring generated by the polynomials of weight $e^{m\alpha}$, where α is the highest root of $\mathfrak{sl}(n, C)$, $m \in \mathbb{Z}$.
We prove that this subring is a polynomial ring in 2 (n -1) generators which are explicitely computed.

Introduction

Let $G = SL(n, C)$; $g = \mathfrak{sl}(n, C)$; $N = \{ A = (a_{ij}) \in M(n, C) : a_{ii} = 1 , a_{ij} = 0$ if $i > j \}$.

We consider the following problem: describe the ring structure of $P[g]^N$, where $P[g]$ are the polynomials on g and the action of N is induced from the adjoint action on g, i.e.: $(n.Q)(x) = Q(n^{-1} x n)$.

It has been proved in [G] that $P[g]^N$ is finitely generated. It is difficult to say more in general. We will restrict ourselves to study a subring of it. Let H be the Cartan subgroup which consists of diagonal matrices. Then the action of H on $P[g]$ is diagonalizable and, as H stabilizes N, it preserves $P[g]^N$ so: $P[g]^N = \bigoplus_{\lambda \in \hat{H}} P[g]^N_\lambda$, where $\lambda : H \to C$ is a homomorphism and $P[g]^N_\lambda = \{p \in P[g]^N : h.p = \lambda(h) p \ \forall \ h \in H\}$.

Let us take the character $\chi : \to C, \chi((a_{ij})) = a_{11} a_n^{-1}$ and consider $\bigoplus_{m=0}^{\infty} P[g]^N_{\chi m}$, it is easy to see that these are exactly the invariants under the group $U = \text{Ker}\, \chi \times N$.

We will prove:

Theorem. $P[g]^U$ *is isomorphic to a polynomial ring in* 2 (n -1) *variables*.

The proof consists of the following steps: first we show that a certain set of 2 (n -1) polynomials are U-invariant and algebraically independent; then we compute the dimension of $P[g]^N_{\chi m}$ as a $P[g]^G$–module and we see that it is the same as the dimension of the polynomials of degree m in n -1 variables.

As we have n -1 polynomials from $P[g]^N_{\chi m}$ that are algebraically independent over $P[g]^G$ we have found a set of free generators of $P[g]^U$.

1. The set of polynomials.

Let $V = (a_{ij})$ the matrix with $a_{ij} = \delta_{1i} \delta_{nj}$, $1 \le i, j \le n$.

On $\mathfrak{sl}(n, C) = \{A \in M(n, C) : \text{tr}\, A = 0\}$, we consider the following polynomials: $\{\text{tr}\, X^i\}_{i=2}^{n}$, $\{\text{tr}\, X^i V\}_{i=1}^{n-1}$. It is known from last century that the first set gives a basis for $P[g]^G$.

The second set consists of N-invariants since :

$$\begin{aligned} \mathrm{tr}\,(n^{-1} X h)^i V &= \mathrm{tr}\; n^{-1} X^i\; n V \\ &= \mathrm{tr}\; X^i\; n V n^{-1} \\ &= \mathrm{tr}\; X^i V . \end{aligned}$$

Furthermore they are U-invariants , since for $h = (a_{ij}) \in H$ we have :

$$\begin{aligned} \mathrm{tr}\,(h^{-1} X h)^i V &= \mathrm{tr}\; h^{-1} X^i\; h V \\ &= \mathrm{tr}\; X^i h V h^{-1} \\ &= \chi(h)\; \mathrm{tr}\, X^i V . \end{aligned}$$

2 . Algebraic independence

To prove that the given 2 (n - 1) polynomials are algebraically independent we restrict them to the space

$$S = \left\{ X \in M(n,C), X = \begin{bmatrix} d_1 & 0 & 0 & \\ x_1 & d_2 & & \\ . & & & \\ . & 1 & d_3 & 0 \\ . & . & . & . \\ x_{n-1} & 1 & 1 & d_n \end{bmatrix} \right\}$$

this is ; the lower triangular matrices whose coefficients are all ones, except for the diagonal and the first column .

It is obvious that the polynomials $\{\mathrm{tr}\, x^i_{|S}\}_{i=2}^{n}$ depend only on the diagonal $d = (d_1,\ldots,d_n)$ and that they are algebraically independent. Also by induction it is easy to prove that tr X^i V are polynomials linear in the $\{x_i\}_{i=1}^{n-1}$ therefore they are algebraically independent if and only if the determinant of the system, which is a polynomial in the $\{d_i\}_{i=1}^{n}$ is different from zero. Furthermore when we set $d_i = 0$, $i = 1, \ldots, n$ we obtain an algebraically independent set of polynomials because $\mathrm{tr}\, X^i V_{|S|\,d=0}$ depends only on $x_1, \ldots, x_{n-i}$ and the coefficient of x_{n-i} is non zero . So the determinant of the system is not zero at d = 0, therefore if we had a relation

$$\Sigma\, c^{\,j_2 \;\; j_n}_{\,i_1 \;\; i_{n-1}} \;\; \mathrm{tr}\, {X^2}^{j_2} \ldots \mathrm{tr}\, {X^n}^{j_n} \;\; \mathrm{tr}\; XV^{i_1} \ldots \mathrm{tr}\; X^{n-1}\, V^{i_{n-1}} = 0$$

restricting to a neighborhood of d = 0 such that the determinant is non zero we would have by algebraically independence of $\{\mathrm{tr}\, X^i V_{|S}\}_{i=1}^{n-1}$ and because $\{\mathrm{tr}\;\; x^i_{|S}\}_{i=2}^{n}$ are constants for fixed d, that :

$$\sum_j c^{\,j_2 \;\; j_n}_{\,i_1 \;\; i_{n-1}} \;\; \mathrm{tr}\, {X^2}^{\,j_1} \ldots \mathrm{tr}\, {X^n}^{j_n} = 0 \;\; \forall\; (i_1, \ldots, 1_{n-1})$$

in this neighborhood, as we are dealing with polynomials, the relation holds everywhere and the fact that $\{\mathrm{tr}\, X^i|_S\}_{i=2}^n$ are algebraically independent implies $c_{i_1\ \ i_{n-1}}^{j_2\ \ j_n} = 0\ \forall\ (i_1\ i_{n-1}),(j_2\ j_n)$.

3. $\dim_{P[g]^G} P[g]_{\chi^m}$.

Let us consider the ring R generated by $Y \subset P[g]$, and consider it as a $P[g]^G$ module, $\{\mathrm{tr}\, x^i\}_{i=2}^n$ generate $P[g]^G$), we will compare $\dim_{P[g]^G} R_{\chi^m}$ with $\dim_{P[g]^G} P[g]_{\chi^m}$ and from the equality we will deduce that $R = P[g]^K$.

Now, $\dim_{P[g]^G} R_{\chi^m}$ is nothing else than P_{n-1}^m = dimension of the space of polynomials of degree m in n-1 variables.

To compute $\dim_{P[g]^G} P[g]_{\chi^m}$ we use Kostant's theorem [K1] which states that:

$P[g] = P[g]^G \otimes H$ and $\dim H_\lambda = m_\lambda(0)$ = multiplicity of the weight 0 in the irreducible representation of g of highest weight λ.

It is easy to see that $\dim_{\mathbb{C}} H_{m\gamma} = \dim_{P[g]^G} P[g]_{\chi^m}$ where γ is the highest root of g (then $\chi = e^\gamma$).

So in order to prove the theorem we must show $m_{k\gamma}(0) = P_{n-1}^k\ \forall\ k \geq 0$. For this we use Kostant's multiplicity formula [K2],

$$m_\lambda(\mu) = \sum_{\sigma \in W} \mathrm{sg}\, \sigma\, K(\delta + \mu - \sigma(\delta + \lambda)$$

where $K(\alpha)$ is Kostant's partition function, i .e . $K(\alpha)$ is the number of ways in wich $-\alpha$ can be written as a sum of positive roots . Here $W = S_n$ is the Weyl group and $\mathrm{sg}\,\sigma$ is the sign of the permutation σ.

In our case we have

$$m_{k\gamma}(0) = \sum_{\sigma \in W} \mathrm{sg}\, \sigma\, K(\delta - \sigma(\delta + k\gamma)$$

To compute it we need the following :

Proposition.

$$\sum_{\sigma \in W} \mathrm{sg}\, \sigma\, K(\delta - \sigma(\delta + k\gamma)) = \sum_{\sigma \in W_\gamma} \mathrm{sg}\, \sigma\, K(\delta - \sigma\delta - k\gamma)$$

where $W_\gamma = \{\tau \in W : \tau\gamma = \gamma\}$

We will prove that $K(\delta - \sigma\delta - k\sigma\gamma) > 0$ implies $\sigma\gamma = \gamma$.

For this we write $W = \bigcup_{i=1}^{N} \sigma_i W_\gamma$, $N = |\Delta|$, $\sigma_i \in W$.

We consider $P = \{\alpha_i\}_{j=1}^{n-1}$ the usual basis of simple roots of the root system Δ, i.e. $\Delta = \{\pm(\alpha_i + \alpha_{i+1} + \quad + \alpha_j) ; 1 \leq i < j \leq n-1\}$, and so the Weyl group is generated by the reflections $r_i\lambda = \frac{\lambda - 2(\lambda_1\ \alpha_i)}{\alpha_i\ \ \alpha_i}\alpha_i$.

It is easy to see that we can take the representatives σ_i to be of the following form: 1) id 2) $r_1\ r_2 \ldots r_{n-1}\ r_{n-2} \ldots r_1$ 3) wr_1 4) wr_{n-1} where $w \in W_\gamma$.

And now, we will check that in cases 2), 3) and 4) the coefficient k_1 in

$\delta - \sigma_i w' \delta - k \sigma_i w' \gamma = \Sigma k_i \alpha_i$ is positive, $\forall\ w' \in W_\gamma$, $\forall\ k \in N$, and so $K(\delta - \sigma\delta - k\sigma\gamma) = 0$ if $\sigma\gamma \neq \gamma$.

Case 2) follows since $\delta - \sigma_i w' \delta - k \sigma_i w' \gamma = \delta \sigma_i w' \delta + k\gamma$ and $\delta - \sigma\delta = \Sigma m_i(\sigma)\alpha_i$ with $m_i(\sigma) \geq 0$.

For the case 3) write:

$$\delta - wr_1\ w'\delta - kwr_1\gamma = \underset{1}{\underline{\delta - w\delta}} + \underset{2}{\underline{w(\delta - r_1\delta)}} + \underset{3}{\underline{wr_1(\delta - w'\delta)}} - \underset{4}{\underline{kw(\alpha_2 + \ldots + \alpha_{n-1})}}$$

And the first and fourth terms don't contain α_1, in the second α_1 has a positive coefficient, and in the third term has a non-negative one. So the sum cannot be written as a sum of negative roots and $K(\delta - wr_1\ w'\delta - kwr_1\delta) = 0$. Case 4) is similar to case 3) and the proposition is proved.

Finally we need the following:

Proposition.

Let W be the Weyl group of a semisimple Lie algebra g and let it be generated by reflections $\{r_i\}_{j=1}^n$ with respect to the hiperplanes defined by the simple roots $\{\alpha_i\}_{j=1}^n$. Take a subsystem Δ_0 generated by $\{\alpha_{i_k}\}_{k=1}^l$ and the corresponding Weyl group $W_0 = <\{r_{i_k}\}_{k=1}^l>$.

Let $\delta = 1/2 \sum_{\alpha\in\Delta^+}\alpha$ then $\sum_{\sigma\in W_0} sg\,\sigma K(\delta - \sigma\delta + \mu) = K_{\Delta\setminus\Delta_0}(\mu)$, where

$K_{\Delta\setminus\Delta_0}(\mu)$ denotes the number of ways in which $-\mu$ can be written as a sum of positive roots of Δ not belonging to Δ_0.

Proof. Let Λ be the root lattice and $\mathbf{Z}[\Lambda]$ the group ring over Λ that is the set of formal sums $\{\sum_{\lambda\in\Lambda} n_\lambda e^\lambda\ ;\ n_\lambda \in \mathbf{Z}\}$ and consider the projection into the identity component e^0.

$$p_1 : \mathbf{Z}[\Lambda] \to \mathbf{Z}, \quad p_1(\Sigma n_\lambda e^\lambda) = n_0 ,$$

then $K(\mu) = p_1(e^{-\mu}\,\Sigma K(\lambda)e^\lambda) = p_1(e^{-\mu}\prod_{\alpha>0}(1 - e^{-\alpha})^{-1})$.

Put $\delta_0 = 1/2\sum_{\alpha\in\Delta_0^+}\alpha$, and notice that $\sigma\delta - \delta = \sigma\delta_0 - \delta_0\ \forall\sigma\in W_0$.

Then Weyl's identity for the subsystem Δ_0 gives

$$\Sigma\ sg\,\sigma e^{\sigma\delta-\delta} = \prod_{\alpha>\Delta_0^+}(1 - e^{-\alpha})^{-1}$$

Now we can prove the proposition:

$$\sum_{\sigma\in W_0} sg\,\sigma K(\delta - \sigma\delta + \mu) = \sum_{\sigma\in W_0} sg\ \sigma\ p_1(e^{-\mu+\sigma\delta-\delta}\prod_{\alpha\in\Delta^+}(1 - e^{-\alpha})^{-1})$$

$$= p_1(\sum_{\sigma\in W_0} sg\,\sigma\ e^{-\mu+\sigma\delta-\sigma}\prod_{\alpha\in\Delta^+}(1 - e^{-\alpha})^{-1})$$

$$= p_1(e^{-\mu}\sum_{\sigma\in W_0} sg\ \sigma e^{\sigma\delta-\delta}\prod_{\alpha\in\Delta_0^+}(1 - e^{-\alpha})^{-1}\prod_{\alpha\in\Delta^+\setminus\Delta_0}(1 - e^{-\alpha})^{-1})$$

$$= p_1 \left(e^{-\mu} \prod_{\alpha \in \Delta^+ \setminus \Delta_0} (1 - e^{-\alpha})^{-1} \right)$$

$$= K_{\Delta \setminus \Delta_0} (\mu)$$

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