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THE EQUIVARIANT INVERSE PROBLEM FOR KLEIN-GORDON-TYPE EXPRESSIONS

M. C. LOPEZ - C. G. SCHIFINI

Abstract

In affirmative, the equivariant inverse problem for Klein-Gordon-type Euler-Lagrange expression is solved.

1. Introduction

It is very well known that the equation describing the scalar field of particles with mass K and spin 0, which is the relativistic counterpart of the Schrödinger equation, is the Klein-Gordon equation

$$(1.1) \qquad g^{ij} \varnothing_{|ij} - K^2 \varnothing = 0 ,$$

where g_{ij} is the metric tensor, $\varnothing$ is a scalar field, and a vertical bar denotes covariant derivative.

This equation can be deduced from a variational principle as follows. If

$$(1.2) \qquad L = L(g_{ij}; \varnothing; \varnothing_{,i})$$

then from a variation of $\varnothing$ we obtain as Euler-Lagrange equation

$$(1.3) \qquad E(L) = - \frac{\partial L}{\partial \varnothing} + \frac{\partial}{\partial x^i} + \left(\frac{\partial L}{\partial \varnothing_{,i}} \right) = 0 ,$$

on choosing

$$(1.4) \qquad L = \frac{1}{2} \sqrt{g} \, (g^{ij} \varnothing_{,i} \varnothing_{,j} + K^2 \varnothing^2)$$

equation (1.3) becomes (1.1).

The equation (1.1) has the following property of covariance: by a transformation of coordinates it changes as a scalar. This property is also possessed by the Lagrangian (1.4).

However, the last assertion is not mandatory since the Lagrangian does not have, in general, any physical meaning, although the Euler-Lagrange expression do have a meaning. The main purpose of this article is to prove that the situation already encountered with the Klein-Gordon equation is found always, i.e., the assumption of the covariance property for the Euler-Lagrange expression implies that the Lagrangian is equivalent to (it has the same Euler-Lagrange expression as) a Lagrangian with the same property. This solves for the affirmative the equivariant inverse problem for Klein-Gordon-type Euler-Lagrange expression [1].

Precisely, we consider a quantity

(1.5) $$H = H(g_{ij}\ ;\ g_{ij,h}\ ;\ \varnothing\ ;\ \varnothing_{,i}\ ;\varnothing_{,ij})$$

such that

(1.6) (i) H is a scalar density
(ii) $H = E(L)$ for L of the type (1.2).

Here we do not assume any covariance property for L with respect to transformations of coordinates. We will prove that (1.5) and (1.6) imply the existence of a scalar density $L_1 = L_1(g_{ij};\ \varnothing\ ;\varnothing_{,j})$ such that $E(L_1) = H$.

2. The equivariant inverse problem

The condition (ii) in (1.6) written in full is

(2.1) $$H = L^{\varnothing} - L^{i;hk}\ g_{hk,i} - L^{i;\varnothing}\ \varnothing_{,i} - L^{i;h}\ \varnothing_{,ih},$$

where $L^{\varnothing} = \partial L/\partial\varnothing$, $L^i = \partial L/\partial\varnothing_{,i}$, and $L^{hk} = \partial L/\partial g_{hk}$

Given any point in the underlying manifold there is a coordinate system $\bar{x} = \bar{x}(x)$ such that, (at the point)

$$B^a_b = \frac{\partial x^a}{\partial \bar{x}^b} = \delta^a_b$$

and so $B = \det(\partial x^i/\partial\bar{x}^j) = 1$, (at the point).

Since H is a scalar density, for this transformation of coordinates we have, (at the point)

(2.2) $$H(g_{hk}\ ;\ g_{hk,i} + B^r_{hi}\ g_{rk} + B^r_{ki}\ g_{rh}\ ;\ \varnothing\ ;\ \varnothing_{,i}\ ;\ \varnothing_{,ih} + B^r_{ih}\ \varnothing_{,r}) =$$
$$= H(g_{hk}\ ;\ g_{hk,i};\ \varnothing\ ;\ \varnothing_{,i}\ ;\ \varnothing_{,ih}),$$

where $B^r_{hi} = \partial^2 x^r/\partial\bar{x}^h\ \partial\bar{x}^i$

From (2.1) and taking into account that is coefficients depend only on g_{ij}, $\varnothing$, and $\varnothing_{,i}$ we deduce that

$$L^{\varnothing} - L^{i;hk}\ (g_{hk,i} + B^r_{hi}\ g_{rk} + B^r_{ki}\ g_{rh}) - L^{i;\varnothing}\ \varnothing_{,i} - L^{i;h}\ (\varnothing_{,ih} + B^r_{ih}\ \varnothing_{,r}) =$$
$$= L^{\varnothing} - L^{i;hk}\ g_{hk,i} - L^{i;\varnothing}\ \varnothing_{,i} - L^{i;h}\ (\varnothing_{,ih}$$

from where, by the symmetry $L^{i;hk} = L^{i;kh}$ we conclude

(2.3) $$B^r_{hi}\ (2L^{i;hk}\ g_{rk} + L^{i;h}\ \varnothing_{,r}) = 0$$

We can suppose that the transformation of coordinates considered satisfies furthermore

$$B^r_{hi} = \frac{1}{2}\ \delta^r_a\ (\ \delta^b_h\ \delta^c_i + \delta^c_h\ \delta^b_i\ )$$

Replacing in (2.3) we obtain

$$(L^{c;bk} + L^{b;ck})\, g_{ak} + L^{b;c}\, \emptyset_{,a} = 0\ .$$

Contracting the last equation with g^{at} it follows

$$L^{c;bt} + L^{b;ct} = -L^{b;c}\, \emptyset^t\ , \tag{2.4}$$

where

$$\emptyset^t = g^{at}\, \emptyset_{,a}\ . \tag{2.5}$$

Changing b with t and c with t in (2.4) we obtain two equations. Adding them and substracting (2.4) it follows that

$$L^{t;cb} = \frac{1}{2}\ [\ L^{b;c}\, \emptyset^t - L^{t;c}\, \emptyset^b - L^{b;t}\, \emptyset^c\ ]\ . \tag{2.6}$$

A straightforward computation proves that

$$L^{i;hk}\, g_{hk,i} + L^{i;j}\, \emptyset_{,ij} = L^{i;j}\, (-\Gamma^t_{ij}\, \emptyset_{,t} + \emptyset_{,ij}) = L^{i;j}\, \emptyset_{|ij}\ , \tag{2.7}$$

where Γ^t_{ij} are the Christoffel symbols corresponding to the metric considered.
Taking into account (2.7) we obtain from (2.1)

$$H = L^{\emptyset} - L^{i;\emptyset}\, \emptyset_{,i} - L^{i;h}\, \emptyset_{|ih}\ . \tag{2.8}$$

Since H is a scalar density we deduce that $\partial H/\partial \emptyset_{,ij} = L^{i;j}$ is a symmetric tensorial density depending on g_{ij}, $\emptyset$, and $\emptyset_{,i}$. Then it is known [2] that there are functions $A = A(\emptyset, \psi)$, $B = B(\emptyset, \psi)$ such that

$$-L^{i;j} = \sqrt{g}\,(A\, g^{ij} + B\, \emptyset^i\, \emptyset^j)\ , \tag{2.9}$$

where

$$\psi = g^{ij}\, \emptyset_{,i}\, \emptyset_{,j} \tag{2.10}$$

Since H and $L^{i;h}\, \emptyset_{|ih}$ are scalar densities we deduce from (2.8) that $L^{\emptyset} - L^{i;\emptyset}\, \emptyset_{,i}$ is a scalar density depending on g_{ij}, $\emptyset$, and $\emptyset_{,i}$. Then it is known [2] that there is a function $C = C(\emptyset, \psi)$ such that

$$L^{\emptyset} - L^{i;\emptyset}\, \emptyset_{,i} = \sqrt{g}\, C\ . \tag{2.11}$$

Then we conclude from (2.8), (2.9), and (2.11) that

$$H = \sqrt{g}\ (C + A\, g^{ij}\, \emptyset_{|ij} + B\, \emptyset^i\, \emptyset^j\, \emptyset_{|ij})\ . \tag{2.12}$$

Since H is an Euler-Lagrange expression it satisfies certain identities [3].

$$\frac{\partial H}{\partial \emptyset_{,ij}} = \frac{\partial H}{\partial \emptyset_{,ij}}\ , \tag{2.13}$$

(2.14)
$$\frac{\partial H}{\partial \varnothing_{,s}} = \frac{\partial}{\partial x^t}\left(\frac{\partial H}{\partial \varnothing_{,st}}\right) ,$$

(2.15)
$$\frac{\partial}{\partial x^s}\left(\frac{\partial H}{\partial \varnothing_{,s}}\right) = \frac{\partial^2}{\partial x^s\, \partial x^t}\left(\frac{\partial H}{\partial \varnothing_{,st}}\right) .$$

Clearly (2.13) is an identity and (2.14) implies (2.15). Let us see what restrictions imposes (2.14) on A, B, and C.

Taking account of (2.5), (2.10) and (2.12) we obtain from (2.14)

(2.16)
$$\frac{1}{2} A g^{st} g^{ij} g_{ij,t} + \frac{1}{2} B \varnothing^s \varnothing^t g^{ij} g_{ij,t} + \frac{\partial A}{\partial \varnothing} \varnothing^s +$$
$$+ \frac{\partial A}{\partial \psi} \varnothing_{,i}\, \varnothing_{,j}\, g^{st} g^{ij}_{,t} + 2 \frac{\partial A}{\partial \psi} \varnothing^i\, \varnothing_{,it} g^{st} + A g^{st}_{,t} +$$
$$+ \frac{\partial B}{\partial \varnothing} \psi\, \varnothing^s + \frac{\partial B}{\partial \psi} \varnothing^s\, \varnothing^t\, \varnothing_{,i}\, \varnothing_{,j}\, g^{ij}_{,t} + B \varnothing^s\, \varnothing_{,it} g^{it} +$$
$$+ B\, \varnothing^s\, \varnothing_{,i}\, g^{it}_{,t} + B \varnothing^t\, \varnothing_{,it} g^{is} + B \varnothing^t\, \varnothing_{,i}\, g^{is}_{,t} +$$
$$- 2 \frac{\partial C}{\partial \psi} \varnothing^s - 2 \frac{\partial A}{\partial \psi} \varnothing^s\, g^{ij}\, \varnothing_{,ij} + 2 \frac{\partial A}{\partial \psi} \varnothing^s\, \varnothing_{,h}\, g^{ij}\, \Gamma^h_{ij} +$$
$$+ A g^{ij}\, \Gamma^s_{ij} - 2 B \varnothing^j \varnothing_{,ij}\, g^{si} + 2 \frac{\partial B}{\partial \psi} \varnothing^s\, \varnothing^i\, \varnothing^j \varnothing_{,h}\, \Gamma^h_{ij} +$$
$$+ 2 B \varnothing^j\, \varnothing_{,h}\, g^{si} \Gamma^h_{ij} + B\, \varnothing^i\, \varnothing^j\, \Gamma^h_{ij} = 0 .$$

Differentiating (2.16) with respect to $\varnothing_{,ij}$ we have

(2.17)
$$2 \frac{\partial A}{\partial \psi} \varnothing^t\, g^{ij} + \frac{1}{2} B \varnothing^i\, \varnothing^{jt} + \frac{1}{2} B \varnothing^j g^{ij} - \frac{\partial A}{\partial \psi} \varnothing^i\, \varnothing^{jt} +$$
$$- \frac{\partial A}{\partial \psi} \varnothing^j\, g^{it} - B \varnothing^t g^{ij} = 0 .$$

Contracting (2.17) with g_{ij} we obtain, for $n \neq 1$, $(2\, \partial A / \partial\psi - B)\, \varnothing^t = 0$, from where we have

(2.18)
$$B = 2 \frac{\partial A}{\partial \psi}$$

It is known that given a point in the underlying manifold there is a coordinate system such that, at the point, $g_{ij,h} = 0$.

Then using (2.18) we deduce from (2.16) that

$$2\,\frac{\partial C}{\partial \psi} - \frac{\partial A}{\partial \varnothing} = 2\,\frac{\partial^2 A}{\partial \psi\,\partial \varnothing} \quad . \tag{2.19}$$

Now, (2.19) is the integrability condition that we need to establish the existence of a scalar $f = f(\varnothing, \psi)$ such that

$$\frac{\partial f}{\partial \varnothing} = C - \frac{\partial A}{\partial \varnothing}\,\psi\,, \qquad \frac{\partial f}{\partial \varnothing} = -\frac{1}{2}\,A \tag{2.20}$$

In this case a straghtforward computation proves that

$$H = E(L_1)\,, \tag{2.21}$$

where $L_1 = \sqrt{g}\; f$ is a scalar density.

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References

[1] ANDERSON, I.M., Ann. Math., 120, 329 (1984).
[2] HORNDESKI G.W. and LOVELOCK, D., Tensor (N.S), 24, 79 (1972).
[3] ANDERSON I.M. and DUCHAMP, T., Am. J. Math.102 (5), 781 (1980).

M. C. LÓPEZ
Departamento de Matemática - Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, 1428 - Buenos Aires, Argentina.

C. G. SCHIFINI
Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)
Buenos Aires, Argentina.

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