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A NOTE ON THE CONMUTATOR OF THE HILBERT TRANSFORM

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Introduction

The purpose of this paper is to give a different proof of a result of S. Bloom on the commutator of the Hilbert transform, see [1]. The idea here, is to imitate the proof of A. P. Calderón for the derivative of the commutator, see [2]. The background for this paper are well known results on the duality of weighted H^1 Hardy spaces, see [3], and certain estimates stated in Lemmas 1, 2 and 3 due to S. Bloom [1]. We believe that Lemma 4 is a contribution to simplify the proofs.

Notations

Let $\omega \geq 0$ be a locally integrable function on R with respect to a measure μ. This ω is said to belong to the class $A_p(d\mu)$ if

$$(\mu(I)^{-1} \int_I \omega \, d\mu)(\mu(I)^{-1} \int_I \omega^{-p/p'} d\mu)^{p/p'} \leq C ,$$

for every interval I. If μ is a doubling measure, i.e. $\mu(2I) \leq c\,\mu(I)$ for every interval I, then, we define

$$M_\mu(f)(x) = \sup_{x \in I} \mu(I)^{-1} \int_I |f(y)| d\mu(y) .$$

It is well-known that if $\omega \in A_p(\mu)$, $1 < p < \infty$, then

$$\left(\int M_\mu(f)^p \omega \, d\mu\right)^{1/p} \leq C_\mu \left(\int |f|^p \omega \, d\mu\right)^{1/p} .$$

Let us denote $m_I g = |I|^{-1} \int_I g(x) dx$, where I is an interval. We say that b belongs to $BMO(\nu)$ if for every interval I

$$\int_I |b(x) - m_I b| \, dx \leq c \int_I \nu(x) \, dx = c\,\nu(I) ,$$

holds. If ω is a weight, we shall say that f belongs to $L^p(\omega)$ if $\left(\int |f(x)|^p \omega(x)dx\right)^{1/p} = \|f\|_{L^p(\omega)} < \infty$. As a general reference we indicate [4].

Statement of the result

We shall prove the following theorems:

Theorem 1 (Commutator theorem of S. Bloom). *Let $\nu \in A_2$ and $\alpha, \beta \in A_2$ such that $\alpha = \nu^p \beta$. Then, if $b \in BMO(\nu)$, the operator*

$$C_b(f)(x) = p.v. \int \frac{b(x) - b(y)}{x - y} f(y)\, dy ,$$

is bounded from $L^p(\alpha)$ into $L^p(\beta)$.

Theorem 2. *Under the same hypotheses of Theorem 1, the operator*

$$R_b^\varepsilon(f)(x) = \int |b(x) - b(y)| \frac{\varepsilon}{(x-y)^2 + \varepsilon^2} |f(y)| dy ,$$

is bounded from $L^p(\alpha)$ into $L^p(\beta)$ with a norm uniformly bounded in ε.

The proofs

We shall need some lemmas.

Lemma 1. *Let $I = (x - \varepsilon, x + \varepsilon)$ and $I_k = (x - \varepsilon 2^k, x + \varepsilon 2^k)$, k a non negative integer. Then, if $\nu \in A_2$ and $b \in BMO(\nu)$, it follows that*

$$|m_I b - m_{I_k} b| \le c\, 2^{k(1-\eta)} (|I_k|^{-1} \int_{I_k} \nu) ,$$

for some $0 < \eta < 1$ depending on ν.

Lemma 2. *If $\omega \in A_p$, there exist $\varepsilon > 0$ such that for all $p' \le r \le p' + \varepsilon$, we have $\omega^{-r/p} \in A_r$.*

Lemma 3. *If $b \in BMO(\nu)$, $\nu \in A_2$ and $\alpha = \nu^p \beta$, α and β in A_p, then, there exists $\varepsilon > 0$ such that for all $p' \le r \le p' + \varepsilon$, we have*

$$|I|^{-1} \int_I |b - m_I b|^r \alpha^{-r/p} \le c |I|^{-1} \int_I \beta^{-r/p} .$$

The proofs of Lemmas 1, 2 and 3 can be found in [1].

Corollary. *The following inequality*

$$(|I_k|^{-1}\int_{I_k}|b-m_Ib|^r\alpha^{-r/p})^{1/r}\le$$
$$c2^{k(1-\eta)}(|I_k|^{-1}\int_{I_k}\beta^{r'/p})^{-1/r'}$$

holds.

Proof. It follows from Lemma 3, making use of Lemmas 1 and 2.

Lemma 4. *If $\beta\in A_p$, there exists $\varepsilon>0$ such that for all r, $p'<r<p'+\varepsilon$, we have $\beta^{1-r'/p}\in A_{p/r'}(\beta^{r'/p}dx)$.*

Proof. By Lemma 2, if $r=p'(1+\delta)$ with δ small enough, we have $\beta^{-r/p}\in A_r$ and if again, we choose δ even smaller we get $\beta\in A_{p_1}$, with $p_1=p/r+1<p$. We have to show that

$$\left(\int_I\beta^{r'/p}\right)^{-1}\left(\int_I\beta^{1-r'/p}\beta^{r'/p}\right)\left(\int_I\beta^{r'/p}\right)^{1-p/r'}\cdot$$
$$\left(\int_I\beta^{-(1-r'/p)/(p/r'-1)}\beta^{r'/p}\right)^{p/r'-1},$$

is bounded by a constant not depending on I. The expression above is equal to

$$\left(\int_I\beta^{r'/p}\right)^{-p/r'}\left(\int_I\beta\right)|I|^{p/r'-1}=$$
$$(|I|^{-1}\int_I\beta)(|I|^{-1}\int_I\beta^{r'/p})^{-p/r'}.$$

Since $\beta^{-r/p}\in A_r$, this expression is bounded by a constant times

$$(|I|^{-1}\int_I\beta)(|I|^{-1}\int_I\beta^{-r/p})^{p/r}.$$

Recalling that $p/r=p_1-1$ and that $\beta\in A_{p_1}$ we get that this is bounded by a constant, as we wanted to show.

As it is well known, see [3], if $f\in L^p(\omega)$, $\omega\in A_p$, $1<p<\infty$, we have

$$f_+(x+it)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(y)}{y-x-it}\,dy\,,$$

and

$$f_-(x-it)=-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(y)}{y-x+it}\,dy\,,$$

$t > 0$, define holomorphic functions on the upper and lower half spaces, respectively. Moreover, the limits for t tending to zero $f_+(x)$ and $f_-(x)$ exist a.e. and $2f(x) = f_+(x) + f_-(x)$ a.e. On the other hand,

$$\sup_{t>0} \|f_\pm(x \pm it)\|_{L^p(\omega)} = \|f_\pm(x)\|_{L^p(\omega)} \leq c_\omega \|f\|_{L^p(\omega)} \cdot$$

We shall denote by D the set of functions f such that

$$|f_\pm(x \pm it)|(1 + x^2 + y^2)^N \leq C_N$$

holds for every non negative integer N. This set is dense in $L^p(\omega)$, see [2].

Proof of Theorem 1. Let us denote by $C_b^\varepsilon f$ the operator

$$C_b^\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{b(x) - b(y)}{x - y} f(y)dy ,$$

where $b \in BMO(\nu)$, $\nu \in A_2$ and $f \in D$. This integral is well defined. Let us consider

$$A_b^{\pm\varepsilon}(f)(x) = \int \frac{b(x) - b(y)}{x - y \mp i\varepsilon} f(y)dy ,$$

where b, ν and f are as above. It is easy to see that

$$|A_b^{\pm\varepsilon} f(x) - C_b^\varepsilon f(x)| \leq \\ c \int |b(x) - b(y)| \frac{\varepsilon}{(x - y)^2 + \varepsilon^2} |f(y)| dy .$$

Therefore, by Theorem 2, the difference $A_b^{\pm\varepsilon} - C_b^\varepsilon$ is a bounded operator form $L^p(\alpha)$ into $L^p(\beta)$. Let $2f(x) = f_+(x) + f_-(x)$. Then, since

$$C_b^\varepsilon f = C_b^\varepsilon f_+ + C_b^\varepsilon f_- ,$$

in order to prove the theorem, it is enough to prove that $A_b^\varepsilon f_+$ and $A_b^\varepsilon f_-$ are suitably bounded. Let us consider $A_b^\varepsilon f_+$. If $g \in D$, we have

$$\int_{-\infty}^{\infty} g(x) A_b^\varepsilon f_+(x) dx = \int g(x) \left(\int \frac{b(x) - b(y)}{x - y - i\varepsilon} f(y) dy \right) dx = \\ - \int \left(\int \frac{b(y) f_+(y)}{x - y - i\varepsilon} dy \right) g(x) dx = \\ - \pi i \int_{-\infty}^{\infty} b(y)\, f_+(y)\, g_+(y + i\varepsilon)\, dy .$$

The holomorphic function $f_+(x + it) g_+(x + it + i\varepsilon)$ satisfies

$$\int_{-\infty}^{\infty} |f_+(x+it)g_+(x+it+i\varepsilon)|\,\nu(x)dx \leq$$
$$\left(\int_{-\infty}^{\infty} |f_+(x+it)|^p\alpha(x)dx\right)^{1/p} \cdot$$
$$\left(\int_{-\infty}^{\infty} |g_+(x+it+i\varepsilon)|^{p'}\beta(x)^{-p'/p}dx\right)^{1/p'} \leq$$
$$c\|f\|_{L^p(\alpha)}\cdot\|g\|_{L^{p'}(\beta^{-p/p'})}\,.$$

Thus, it belongs to $H^1(\nu)$. Therefore, by the duality between $H^1(\nu)$ and $BM0(\nu)$, see [3], we get

$$\left|\int_{-\infty}^{\infty} g\,A_b^{\varepsilon}f\,dx\right| \leq C\,\|b\|_{BMO(\nu)}\|f\|_{L^p(\alpha)}\|g\|_{L^p(\beta^{-p/p'})}\,,$$

where C does not depend on ε.

A similar argument gives the same estimate for $A_b^{-\varepsilon}f_-$. Thus, $\|C_b^{\varepsilon}f\|_{L^p(\beta)} \leq c\,\|f\|_{L^p(\alpha)}$ and taking limits for ε tending to zero the theorem follows.

Proof of Theorem 2. Let $g \in L^{p'}(\beta^{-p/p'})$. Then, if $I_k = (x-\varepsilon 2^k, x+\varepsilon 2^k)$, we have

$$\int |g(x)|\left(\int |b(x)-b(y)|\frac{\varepsilon}{(x-y)^2+\varepsilon^2}\,|f(y)|dy\right)dx$$
$$\leq \int |f(y)|\left(\int |b(x)-m_{I_0}b|\frac{\varepsilon}{(x-y)^2+\varepsilon^2}\,|g(x)|dx\right)dy$$
$$+\int |g(x)|\left(\int |b(y)-m_{I_0}b|\frac{\varepsilon}{(x-y)^2+\varepsilon^2}\,|f(y)|dy\right)dx$$
$$= I_1 + I_2\,.$$

Let us consider I_2. We have

$$\int |b(y)-m_{I_0}b|\frac{\varepsilon}{(x-y)^2+\varepsilon^2}\,|f(y)|dy \leq$$
$$c\sum_{k=0}^{\infty} 2^{-k}|I_k|^{-1}\int_{I_k} |b(y)-m_{I_0}b|\,|f(y)|dy\,.$$

By Hölder's inequality and the corollary to Lemma 3, the right hand side is bounded by

$$c\sum_{k=0}^{\infty} 2^{-k}2^{k(1-\eta)}(|I_k|^{-1}\int_{I_k}\beta^{r'/p})^{-1/r'}(|I_k|^{-1}\int_{I_k}|f\nu|^{r'}\beta^{r'/p})^{1/r'}$$
$$\leq C(\sum_{k=0}^{\infty}2^{-k\eta})M_{\beta^{r'/p}}(|f\nu|^{r'})(x)^{1/r'}\,.$$

Thus

$$I_2 \le c \int |g(x)| M_{\beta^{r'/p}}(|f\nu|^{r'})(x)^{1/r'}\,dx$$
$$\le c\left(\int |g(x)|^{p'}\beta^{-p/p'}\right)^{1/p'}\left(\int M_{\beta^{r'/p}}(|f\nu|^{r'})^{p/r'}\beta\right)^{1/p}.$$

Since $p > r'$ and, by Lemma 4, $\beta = \beta^{1-r'/p}\beta^{r'/p}$ with $\beta^{1/r'/p} \in A_{p/r'}(\beta^{r'/p}dx)$, we have

$$I_2 \le c\|g\|_{L^{p'}(\beta^{-p'/p})}\|f\|_{L^p(\alpha)}\,.$$

For I_1, taking into account that $\beta^{-p'/p} = \nu^{p'}\alpha^{-p'/p}$, we get the same estimate as for I_2. This ends the proof of the theorem.

References

[1] BLOOM S., *A commutator theorem and weighted BMO,* Trans. Amer. Math. Soc. 292 (1985), 103-122.

[2] CALDERON, A. P., *Commutators of singular integral operators,* Proc. Nat. Acad. Sci. USA 53 (1965), 1092-1099.

[3] GARCIA-CUERVA, J., *Weighted H^p spaces,* Dissert. Mathematicae 162, Warszawa 1979.

[4] GARCIA-CUERVA, J. and RUBIO DE FRANCIA J. L., *Weighted Norm Inequalities and Related Topics,* Mathematics Studies 116, North-Holland, 1985.

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