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REMARKS ON SOME NONLINEAR INITIAL BOUNDARY VALUE PROBLEMS IN HEAT CONDUCTION

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1. Introduction

This paper is concerned with the following initial-boundary value problems for the one-dimensional (normalized) heat equation for a semi-infinite material

$$(1)\qquad \begin{cases} i) \ u_{xx} - u_t = F(u_x(0,t)) \ \ in \ \ 0 < x < \infty, \quad 0 < t < T \\ ii) \ u(x,0) = h(x), \quad x > 0 \\ iii) \ u(0,t) = 0, \quad 0 < t < T \end{cases}$$

and for a finite slab:

$$(2)\qquad \begin{cases} i) \ u_{xx} - u_t = F(u_x(0,t)) \ \ in \ \ 0 < x < 1, \quad 0 < t < T \\ ii) \ u(x,0) = h(x), \quad 0 < x < 1 \\ iii) \ u(0,t) = 0, \quad 0 < t < T \\ iv) \ u(1,t) = f(t), \quad 0 < t < T \end{cases}$$

where $u = u(x,t)$ denote the temperature distribution (the unknown), x and t the spatial and time coordinate respectively; T is a given constant and the data functions h and f represent initial and boundary conditions. F denote a sink or source of heat energy, uniform in x.

The preceding one may be thought as mathematical models of controled temperature distributions in isotropic mediums.

In (1), results on existence, uniqueness of solution and asymptotic behavior have been proved for (1) and (2).

In section §2 and §3 some results on the behavior of the solution and explicit formulas for special cases are obtained for the problems (1) and (2) respectively.

2. Results on problems (1)

We consider problem (1) for the temperature u=u(x,t). Then, let the function V=V(t) be defined by

$$V(t) = u_x(0,t), \quad 0 < t < T \tag{2.1}$$

Taking into account $[V_i]$, the following integral representation for the solution u of (1) can be written

$$u(x,t) = \int_0^\infty G(x,t,\xi,0)\, h(\xi)\, d\xi - \int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) F(V(\tau))\, d\tau \tag{2.2}$$

where

$$V(t) = V_o(t) + \int_0^t H(t,\tau,V(\tau))\, d\tau \tag{2.3}$$

$$V(t) = \frac{1}{2\sqrt{\pi}} \frac{1}{t^{3/2}} \int_0^\infty \xi\, exp\left(-\frac{\xi^2}{4t}\right) h(\xi)\, d\xi \tag{2.4}$$

$$H(t,\tau,V) = -\frac{F(V)}{\sqrt{t-\tau}} \tag{2.5}$$

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau) \tag{2.6}$$

$$K(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right), \quad t > \tau \tag{2.7}$$

Let us consider for function $F = F(V)$ and $h = h(x)$, in problem (1), the following hypothese:

$$(H_1)\begin{cases} i)\; F = F(V) \text{ is a continuos function for all } V \in R, \text{ wich satisfies } F(0) = 0, \\ ii)\; |F(V_1) - F(V_2)| \leq \alpha |V_1 - V_2| \quad with\ \alpha = const > 0 \end{cases}$$

$$(H_2)\begin{cases} i)\; h \in C^1[0,+\infty] \quad with\, h(0) = 0, \\ ii)\; h,\, h' \in L^\infty(0,+\infty) \end{cases}$$

Let us denote by problem (1bis) the problem (1) with data $\overline{F}$ and $\overline{h}$ respectively in conditions (1i) and (1ii).

Then, we obtain the following result on the continuos dependence upon the data.

Theorem 1. *Let u=u(x,t) and $\overline{u} = \overline{u}(x,t)$ be the solutions of problems (1) and (1bis) respectively under the assumptions (H_1) and (H_2) for data F, $\overline{F}$ and h, $\overline{h}$. If the functions F and $\overline{F}$ satisfy the additional hypothese.*

$$(2.8)\qquad |F(V) - \overline{F}(\overline{V})| \leq \beta\,|V - \overline{V}|, \quad \textit{with} \quad \beta = const. \geq 0$$

then we obtain the inequality:
(2.9)
$$|u(x,t) - \overline{u}(x,t)| \leq \|h - \overline{h}\|_\infty + \beta\, T(1 + 2\beta\sqrt{T})exp(\pi\beta^2 T)\,\|h' - \overline{h}'\|_\infty, \ \textit{for } x \leq 0, \quad 0 \leq t \leq T$$

Proof. To begin with, we remark that the existence and uniqueness of solution for the problems (1) and (1bis) follows from hypothese (H_1) and (H_2), [V_i].

If we define $\overline{V}(t) = \overline{u}_x(0,t)$, from expressions (2.3)-(2.5), we obtain
(2.10)
$$\begin{aligned} V(t) - \overline{V}(t) &= \frac{1}{2\sqrt{\pi}t^{3/2}} \int_0^\infty \xi\, exp\left(-\frac{\xi^2}{4t}\right) [h(\xi) - \overline{h}(\xi)]\, d\xi + \int_0^t \frac{F(\overline{V}(\tau)) - F(V(\tau))}{\sqrt{t-\tau}}\, d\tau \\ &= \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} exp\left(-\frac{\xi^2}{4t}\right) [h'(\xi) - \overline{h}'(\xi)]\, d\xi + \int_0^t \frac{F(\overline{V}(\tau)) - F(V(\tau))}{\sqrt{t-\tau}}\, d\tau \end{aligned}$$

By using (2.8), we have

$$(2.11)\qquad |V(t) - \overline{V}(t)| \leq \|h' - \overline{h}'\|_\infty + \beta \int_0^t \frac{|V(\tau) - \overline{V}(\tau)|}{\sqrt{t-\tau}}\, d\tau$$

On the other hand, by the Gronwall's inequality [Ca] applied to (2.11), we deduce

$$(2.12)\qquad \|V - \overline{V}\|_T \leq (1 + 2\beta\sqrt{T})\, exp(\pi\beta^2 T)\, \|h' - \overline{h}'\|_\infty$$

where $\|g\|_t$ denote

$$(2.13)\qquad \|g\|_t = \max_{0 \leq t \leq \tau} |g(\tau)|$$

Now, from (2.2), (2.8) and (2.12), we obtain

$$(2.14)\qquad \begin{aligned} |u(x,t) - \overline{u}(x,t)| &\leq \int_0^{+\infty} G(x,t,\xi,0)\, |h(\xi) - \overline{h}(\xi)|\, d\xi + \\ &\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) |\overline{F}(\overline{V}(\tau)) - F(V(\tau))|\, d\tau \leq \\ &\leq erf\left(\frac{x}{2\sqrt{t}}\right) \|h - \overline{h}'\|_\infty \beta \int_0^t |V(\tau) - \overline{V}(\tau)|\, d\tau, \end{aligned}$$

that is (2.9).

If we consider the special case

(2.15) $$\overline{F}(\overline{V})) = d\overline{V}, \quad with \quad d = const. > 0$$

(2.16) $$h = \overline{h} \in C^{\circ}[0, +\infty],$$

(2.17)
$V_o = V_o(t)$, defined by (2.4), is a non-decreasing function which verifies $V_o(0) > 0$

we obtain the following comparison result.

Theorem 2. *Under the conditions (2.15)-(2.17), if the function $F = F(V)$ verifies hypothese (H_1) and furthermore*

(2.18) $$V\,F(V) > 0, for\ all\ V \in R,$$

(2.19) $$\alpha < d$$

then there exists $t_1 > 0$ such that

(2.20) $$\overline{u}(x,t) \leq u(x,t), \quad 0 \leq x, \quad 0 \leq t \leq t_1$$

where u and $\overline{u}$ are the solutions of problem (1) and (1bis) with data F, h and $\overline{F}, \overline{h} = h$ respectively.

Proof. Owing to (2.15)-(2.18) and [V_i, lemma 2], it follows that $\left(V(t) = u_x(0,t), \quad \overline{V}(t) = \overline{u}_x(0,t)\right)$:

(2.21) $$V(t) > 0, \quad \overline{V}(t) > 0, \quad \forall t \in (0,T)$$

Taking into account (2.2)-(2.5), (2.15) and (2.16), we have

(2.22) $$\overline{u}(x,t) - u(x,t) = \int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right)[F(V(\tau)) - d\overline{V}(\tau)]\,d\tau,$$

(2.23) $$\overline{V}(t) - V(t) = \int_0^t \frac{F(V(\tau)) - d\overline{V}(\tau)}{\sqrt{t-\tau}}$$

From (H_1) and (2.21) we obtain

(2.24) $$F(V) \leq \alpha V$$

and then

(2.25) $$\overline{V}(t) - V(t) \leq d\int_0^t \frac{V(\tau) - \overline{V}(\tau)}{\sqrt{t-\tau}}\,d\tau, \quad 0 < t \leq T$$

It is clear, from (2.25) and continuity properties for $\psi_t = \overline{V}(t) - V(t)$, with $\psi(0) = 0$, that there exists $t_o \in (0, t)$ such that

$$\overline{V}(\tau) \leq V(\tau), \quad 0 < \tau < t_o \tag{2.26}$$

Moreover, from (2.23), (2.26) and the fact that $\phi(t) = F(V(t)) - d\overline{V}(t)$ verifies $\phi(0) \leq 0$, we deduce that there exists $t_1 \in (0, t_o)$, such that

$$F(V(t)) - d\overline{V}(t) \leq 0, \quad 0 < t < t_1 \tag{2.27}$$

that is (2.20) from (2.22).

Remark 1. The comparison result (2.20) can be also obtained by using the maximun principle taking into account the inequality (2.27). Let $U = U(x,t)$ and $W = W(x,t)$ be defined by

$$U(x,t) = \overline{u}(x,t) - u(x,t), \quad x > 0, \quad 0 < t < t_1, \tag{2.28}$$

$$W(x,t) = U_x(x,t), \quad x > 0, \quad 0 < t < t_1 \tag{2.29}$$

They satisfy the following problems

$$\begin{cases} U_{xx} - U_t = d\overline{V} - F(V) \geq 0, & x > 0, \quad 0 < t < t_1, \\ U(x,0) = 0, & x \geq 0, \\ U(0,t) = 0, & 0 < t < t_1 \end{cases} \tag{2.30}$$

$$\begin{cases} W_{xx} - W_t = 0, & x > 0, \quad 0 < t < t_1, \\ W(x,0) = 0, & x \geq 0, \\ W_x(0,t) = d\overline{V}(t) - F(V(t)) \geq 0, & 0 \leq t \leq t_1. \end{cases} \tag{2.31}$$

Then, we obtain

$$W(x,t) \leq 0, \qquad U(x,t) \leq 0, \quad x \geq 0, \quad 0 \leq t \leq t_1. \tag{2.32}$$

Remark 2. If we suppose hypothesis (H_2) with

$$h'(x) = \overline{h}'(x) \geq 0, \quad x \geq 0 \tag{2.33}$$

instead of (2.16) in *Theorem* 2, we deduce the inequality

$$0 \leq \overline{u}(x,t) \leq u(x,t), \quad x \geq 0, \quad 0 \leq t \leq t_1, \tag{2.34}$$

by using the Lemma 3 of $[V_i]$.

If, in problem (1), we suppose that

$$F(V) = \alpha V, \quad with \quad \alpha = const. > 0, \tag{2.35}$$

$$(2.36)\qquad \begin{cases} h \in C^1[0,+\infty] \quad with \quad h(0)=0, \\ h=h(x) \quad and \quad h'=h'(x) \quad are \quad o\left(e^{a_o x}\right) \text{ for x large for some } a_o>0, \end{cases}$$

then, by using the Läplace transformation, we obtain the explicit solution.

Theorem 3. *Under assumptions (2.35) and (2.36), the solution of problem (1) is given by*

$$(2.37)\qquad u(x,t)=\int_0^{+\infty} G(x,t,\xi,0)\,h(\xi)\,d\xi-\alpha\int_0^t V(\tau)\,erf\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau$$

where

$$(2.38)\quad V(t)=V_o(t)-\alpha\int_0^t\left[\frac{1}{\sqrt{t-\eta}}-\alpha\,\pi\,exp(\pi\alpha^2(t-\eta))\,erfc(\alpha\sqrt{\pi(t-\eta)})\right]V_o(\eta)\,d\eta,$$

and V_o is explicited in (2.4).

Proof. From (2.3), (2.5) and (2.35) we obtain the following second kind Volterra integral equation for V=V(t)

$$(2.39)\qquad V(t)=V_o(t)-\alpha\int_0^t R(t,\tau)\,V(\tau)\,d\tau$$

where the kernel $R=R(t,\tau)$ is given by

$$(2.40)\qquad R(t,\tau)=\frac{1}{\sqrt{t-\tau}},\quad t>\tau>0$$

which depends only on the difference of the two arguments t and τ, and it is singular at $t=\tau$.

From elementary calculations, we have

$$(2.41)\qquad V_o(t)=\frac{1}{\sqrt{\pi t}}\int_0^{+\infty} exp\left(-\frac{\xi^2}{4t}\right)h'(\xi)\,d\xi$$

and so that for t large $(M,l>0)$

$$(2.42)\qquad \begin{cases} |V_o(t)|\le M\,exp(lt) \\ |V(t)|\le(1+2\alpha\sqrt{t})\,|V_o(t)|\,exp(\pi\alpha^2 t) \end{cases}$$

that is

$$(2.43)\qquad |V(t)|\le N\,exp(mt)$$

with $\pi\alpha^2+l<m$.

On the other hand, let $W=W(t-\tau)$ be the resolvent kernel of (2.39), that is

$$(2.44)\qquad V(t)=V_o(t)-\alpha\int_0^t W(t-\tau)\,V_o(\tau)\,d\tau$$

If s denote the one-sided Laplace integral transform variable, for $Re(s) > m$, we deduce

$$v(s) = \frac{v_o(s)}{1 + \alpha y(s)}, \quad v(s) = v_o(s)(1 - \alpha Y(s)) \tag{2.45}$$

that is

$$Y(s) = \frac{y(s)}{1 + \alpha y(s)} = \frac{\sqrt{\pi}}{\alpha\sqrt{\pi} + s} \tag{2.46}$$

where

$$\begin{cases} v_o(s) = \mathcal{L}(V_o(t)), & v(s) = \mathcal{L}(V(t)) \\ y(s) = \mathcal{L}(R(t)) = \dfrac{\sqrt{\pi}}{\sqrt{s}}, & Y(s) = \mathcal{L}(W(t)) \end{cases} \tag{2.47}$$

and $\mathcal{L}$ denotes the Laplace transformation.

By using the inversion formula we obtain that.

$$W(t) = \frac{1}{\sqrt{t}} - \alpha\,\pi\; exp(\pi\alpha^2 t)\, erfc(\alpha\sqrt{\pi t}) \tag{2.48}$$

that is (2.38), and so that (2.37).

Remark 3. We can verify directly that (2.37) and (2.38) is the explicit solution of problem (1) for the particular case (2.35) and (2.36). In fact, if we put (2.38) in (2.3) it is sufficient to check that

$$\int_\eta^t \frac{exp(\pi\alpha^2(\tau - \eta))}{\sqrt{t-\tau}}\, erfc(\alpha\sqrt{\pi(\tau-\eta)})\, d\tau = \tag{2.49}$$

$$\frac{1}{\alpha} - \frac{1}{\alpha}\, exp(\pi\alpha^2(t-\eta))\, erfc(\alpha\sqrt{\pi(t-\eta)}), \quad 0 < \eta < t.$$

By using that $erfc(x) = 1 - erf(x)$ and the function $Z = Z(x)$, defined by

$$Z(x) = 2\int_0^x exp(\pi\alpha^2(x^2 - y^2))\, erf(\alpha\sqrt{\pi(x^2 - y^2)})\, dy, \quad x > 0 \tag{2.50}$$

which verifies the Cauchy problem

$$\begin{cases} Z'(x) = 2\pi\alpha^2 x Z(x) + 2\pi\alpha x \\ Z(0) = 0 \end{cases} \tag{2.51}$$

whose solution is given by

$$Z(x) = \frac{1}{\alpha}(exp(\pi\alpha^2 x^2) - 1) \tag{2.52}$$

the equality (2.49) is obtained.

3. Results on problems (2)

Let the function V=V(t) be defined by (2.1). Such as it was pointed out in [V_i] if u=u(x,t) satisfies the problem (2), the following integral representation can be written

$$(3.1)\ u(x,t) = xf(t) + \int_0^1 K_o(x,\xi,t)\,[h(\xi) - f(0)\xi]\,d\xi + \int_0^t K_1(x,t-\tau)\,F(V(\tau))\,d\tau + \int_0^t K_2(x,t-\tau)\,\dot{f}(\tau)\,d\tau$$

where

$$V(t) = \overline{V}_o(t) + \int_0^t \overline{H}(t,\tau,V(\tau))\,d\tau \tag{3.2}$$

$$\overline{V}_o(t) = f(t) + \int_0^1 \overline{K}_o(0,\xi,t)\,[h(\xi) - f(0)\xi]\,d\xi \tag{3.3}$$

$$\overline{H}(t,\tau,V(\tau)) = \overline{K}_1(0,t-\tau)\,F(V(\tau)) + \overline{K}_2(0,t-\tau)\,\dot{f}(\tau) \tag{3.4}$$

being K_o, K_1, K_2, $\overline{K}_0$, $\overline{K}_1$, $\overline{K}_2$ explicited in the *Appendix*.

Let us denote by problem (2 bis) the problem (2) with data $\hat{F}$, $\hat{h}$, $\hat{f}$.

For functions F, h, $\hat{F}$, $\hat{h}$ and $\hat{f}$ in problems (2), (2 bis) we shall suppose the following assumptions

(A_1) [the same as (H_1) of problem (1)] and furthemore (2.18)

$$(A_2)\qquad \begin{cases} i) h,\hat{h} \in C^o[0,\infty) \\ ii) f,\hat{f} \in C^1[0,T] \end{cases}$$

$$(A_3)\qquad \hat{F}(\hat{V}) = a\hat{V}, \quad a > 0 \quad \overline{V}_o(0) > 0$$

Then, we obtain the following comparison result

Theorem 4. *Under the hypothese (H_1), (A_2), (A_3), if the following condition is satisfied*

$$\alpha < a \tag{3.5}$$

with α introduced by ii) of (H_1), there exists $t_o > 0$ such that

$$u(x,t) \geq \hat{u}(x,t), \quad 0 < x < 1, \quad 0 \leq t \leq t_o \tag{3.6}$$

where u and $\hat{u}$ are the solutions of problem (2) and (2bis) respectively.

Proof. In a similar way as in *Theorem* 2 of ≀ 2, we find

$$u(x,t) - \hat{u}(x,t) = \int_0^t K_1(x,t-\tau)\,[F(V(\tau)) - a\hat{V}(\tau)]\,d\tau \tag{3.7}$$

$$V(t) - \hat{V}(t) = \int_0^t (-\overline{K}_1(0,t-\tau))\,[a\hat{V} - F(V)]\,d\tau \tag{3.8}$$

where $\hat{V}(t) = \hat{u}_x(0,t)$

Since $V_o(0) > 0$, from the continuity of V(t) on t=0 it follows that there exists $t_2 > 0$ such that

$$V(t) > 0, \quad 0 < t < t_2 \tag{3.9}$$

Therefore, taking into account (2.18) and (3.5) in equation (3.8) we obtain

$$\frac{\hat{V}(t) - V(t)}{a} < \int_0^t [-K_1(0,t-\tau)]\,[V(\tau) - \hat{V}(\tau)]\,d\tau \tag{3.10}$$

from which, by similar arguments used in proving (2.20), owing to the continuity of the function $K_1(x,t-\tau)$ an the fact that

$$K_1(x,0) = \begin{cases} -\dfrac{1}{2} & if \quad 0 < x \leq 1 \\ \dfrac{1}{2} & if \quad -1 \leq x < 0 \end{cases} \tag{3.11}$$

follows the inequality announced by (3.6).

If, in problem (2), we suppose that

$$F(V) = dV, \quad with \quad d = cont. > 0 \tag{3.12}$$

$$h \in C^o{}_{[0,1]} \tag{3.13}$$

$$f \in C^1{}_{[0,T]}, \quad f^{(n)} = o(e^{b_o t}), \text{ for t large, n=0,1 for some } b_o > 0 \tag{(3.14)}$$

then, by using the Laplace transformation, the explicit solution can be obtained.

Theorem 5. *Under assumptions (3.12), (3.13) and (3.14) the solution of problem (2) is given by*

$$\begin{aligned} \hat{u}(x,t) =& x\,f(t) + \int_0^1 K_o(x,\xi,t)\,[h(\xi) - f(0)\xi]\,d\xi \\ &+ d\int_0^t K_1(x,t-\tau)\,[\hat{V}_o(\tau) - d\int_0^t \Theta(\tau-\eta)\hat{V}_o(\eta)\,d\eta]\,d\tau \end{aligned} \tag{3.15}$$

where

$$(3.16)\qquad \hat{V}_o(t) = f(t) + \int_0^t \overline{K}_o(0,\xi,t)\,[h(\xi) - f(0)\xi]\,d\xi + \int_0^t \overline{K}_2(0,t-\tau)\,\dot{f}(\tau)\,d\tau$$

$$(3.17)\qquad \Theta(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \varphi(s)\,e^{st}\,ds$$

$$(3.18)\qquad \varphi(s) = \frac{\tan(\frac{\sqrt{s}}{2})}{4 + d\dfrac{\tan(\frac{\sqrt{s}}{2})}{(\frac{\sqrt{s}}{2})}}$$

Proof. The proof is similar to the preceding one for *Theorem* 3 and therefore we omit detail here.

Now, the Volterra integral equation is

$$(3.18)\qquad \hat{V}(t) = \hat{V}_o(t) + d\int_0^t \overline{K}_1(0,t-\tau)\,\hat{V}(\tau)\,d\tau$$

where $\hat{V}_o(t)$ is given by (3.16).

Appendix

$$K_o = 2\sum_{k=1}^{\infty} e^{-k^2\pi^2 t}\,\sin(k\pi x)\,\sin(k\pi\xi)$$

$$K_1 = \sum_{k=1}^{\infty} a_k\,e^{-k^2\pi^2(t-\tau)}\,\sin(k\pi x);\quad a_k = -\frac{1}{k\pi}[1-(-1)^k]$$

$$K_2 = \sum_{k=1}^{\infty} b_k\,e^{-k^2\pi^2(t-\tau)}\,\sin(k\pi x)\quad b_k = \frac{1}{k}(-1)^k$$

$$\overline{K}_o(0,\xi,t) = \frac{dK_o}{dx}\quad at\quad x=0$$

$$\overline{K}_1(0,t-\tau) = \frac{dK_1}{dx}\quad at\quad x=0$$

$$\overline{K}_2(0,t-\tau) = \frac{dK_2}{dx}\quad at\quad x=0$$

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