Revista de la
Unión Matemática Argentina
Vol. 35 1990

ON THE COMPOSITION FORMULAS OF THE SOLUTIONS OF THE ULTRAHYPERBOLIC AND KLEIN-GORDON OPERATORS

SUSANA ELENA TRIONE

Abstract

Let $t = (t_1, t_2, \ldots, t_n)$ be a point of R^n. We shall write $t_1^2 + \cdots + t_\mu^2 - t_{\mu+1}^2 - \cdots - t_{\mu+\nu}^2 = u$, $\mu + \nu = n$. We put, by definition, $R_\alpha(u) = \dfrac{u^{\frac{\alpha-2}{2}}}{K_n(\alpha)}$; here α is a complex parameter, n the dimension of the space and the constant $K_n(\alpha)$ is defined by (I,2). $R_\alpha(u)$ is the Marcel Riesz' ultrahyperbolic kernel.

The distributional kernel $R_\alpha(u)$ share many properties with the Riemann-Liouville kernel of which they are n-dimensional ultrahyperbolic analogues. In this paper we prove the following composition formula: $R_\alpha * R_\beta(u) = R_{\alpha+\beta}(u)$, α and $\beta \in C$ (cf. form. (II,6)).

We remark that this formula has been proved by Nogin (cf. [1]), by a completely different method.

In paragraph III we put, by definition,
$$W_\alpha(u,m) = \frac{(m^{-2}u)^{\frac{\alpha-n}{4}}}{\pi^{\frac{n-2}{2}} 2^{\frac{\alpha+n-2}{2}} \Gamma(\frac{\alpha}{2})} J_{\frac{\alpha-n}{2}}(m^2u)^{\frac{1}{2}};$$
here α is a complex parameter, m a real nonnegative number and n the dimension of the space. $W_\alpha(u,m)$, which is an ordinary function if Re $a \geq n$, is an entire distributional function of α. $W_\alpha(u,m)$ is the ultrahyperbolic solution of the Klein-Gordon operator. We shall evaluate the composition formula $W_\alpha * W_\beta(u,m) = W_{\alpha+\beta}(u,m)$, $\alpha, \beta \in C$. (cf. form. (IV,6)). The particular case $\mu = 1$ of this formula was proved in [2] by a completely different manner.

I. The Marcel Riesz' ultrahyperbolic kernel.

Let $t = (t_1, t_2, \ldots, t_n)$ be a point of R^n. We shall write
$$t_1^2 + \cdots + t_\mu^2 - t_{\mu+1}^2 - \cdots - t_{\mu+\nu}^2 = u, \quad \mu + \nu = n.$$

By Γ_+ we designate the interior of the forward cone: $\Gamma_+ = \{t \in R^n / t_1 > 0, u > 0\}$. We consider the following family of functions introduced by Nozaki (cf. [3], p. 72):

$$\text{(I,1)} \qquad R_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } t \in \Gamma_+, \\ 0 & \text{if } t \notin \Gamma_+. \end{cases}$$

Here α is a complex parameter, n the dimension of the space.

The constant $K_n(\alpha)$ is defined by

$$\text{(I,2)} \qquad K_n(\alpha) = \pi^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-\mu}{2}\right)\Gamma\left(\frac{\mu-\alpha}{2}\right)},$$

μ is the number of positive terms of $u = t_1^2 + \cdots + t_\mu^2 - t_{\mu+1}^2 - \cdots - t_{\mu+\nu}^2$, $\mu + \nu = n$.

$R_\alpha(u)$ which is an ordinary function of α if Re $\alpha \geq n$, is a distributional function of α.

We call $R_\alpha(u)$ the ultrahyperbolic kernel of Marcel Riesz.

The distributional kernel $R_\alpha(u)$ share many properties with the one-dimensional Riemann-Liouville kernel of which they are n-dimensional ultrahyperbolic analogues. The following formula have been proved in [4]: $\Box R_{\alpha+2}(u) = R_\alpha(u)$, where $\Box$ is the n-dimensional ultrahyperbolic operator defined by $\Box = \frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_\mu^2} - \frac{\partial^2}{\partial t_{\mu+1}^2} - \cdots - \frac{\partial^2}{\partial t_{\mu+\nu}^2}$, $\mu + \nu = n$; $R_{-2k}(u) = \Box^k \delta$, $k = 1, 2, \ldots$; Ro $(u) = \delta$; $\Box^k R_{2k}(u) = \delta$.

II. The composition formula $R_\alpha * R_\beta(u) = R_{\alpha+\beta}(u)$.

We know from formula (III,9), p.9 of [4] that

$$\text{(II,1)} \qquad R_{-2k}(u) = \Box^k \delta, \quad k = 0, 1, \ldots;$$

which is the n-dimensional ultrahyperbolic correlative of the n-dimensional formula

$$\left.\frac{x_+^{\alpha-1}}{\Gamma(\alpha)}\right|_{\alpha=-n} = \delta^{(n)}(x).$$

We shall evaluate $R_\alpha * R_\beta(u)$. First we observe that this convolution exists because all $R_\alpha(u)$ have their support on the forward cone Γ_+.

We begin by evaluating $R_\alpha * R_{-2k}$. Taking into account (II,1), we have

$$\text{(II,3)} \qquad R_\alpha * R_{-2k} = R_\alpha * \Box^k \delta = \Box^k R_\alpha.$$

From formula (V,2), p.11 of [4], we know that

$$\text{(II,4)} \qquad \Box^k R_\alpha(u) = R_{\alpha-2k}(u),$$

$k = 0, 1, \ldots,$.

This last result coincides, when $\mu = 1$, with (II,3;36), p.58 of [2], and is due to Marcel Riesz.

Then from (II,3) and (II,4), we have

$$R_\alpha * R_{-2k}(u) = R_{\alpha-2k}(u), \tag{II,5}$$

$\alpha \in C$, $k = 0, 1, \ldots$.

Finally, by appealing to the principle of the analytical in continuation, we have

$$R_\alpha * R_\beta(u) = R_{\alpha+\beta}(u), \tag{II,6}$$

$\alpha, \beta \in C$.

The formula (II,6) is due to Nogin (cf. [1]), which has been proved by a completely different method.

We shall use other method to prove (II,6). This is based in the symbolic method due to Marcel Riesz.

In fact we have, formally, from (II,1), that

$$R_\beta = \Box^{-\beta/2}\delta. \tag{II,7}$$

Therefore

$$\begin{aligned} R_\alpha * R_\beta(u) &= R_\alpha * \Box^{-\beta/2}\delta \\ &= \Box^{-\beta/2} R_\alpha(u). \end{aligned} \tag{II,8}$$

Finally, from (II,4) we obtain

$$R_\alpha * R_\beta(u) = R_{\alpha+\beta}(u), \tag{II,9}$$

$\alpha, \beta \in C$.

The complete justification of (II,8) is given by S. E. Trione ([5], pp. 16, 17 and 18).

III. The elementary retarded, ultrahyperbolic solution of the Klein-Gordon operator, iterated k-times.

We shall consider the following functions:

$$W_\alpha(u,m) = \begin{cases} \dfrac{(m^{-2}u)^{\frac{\alpha-n}{4}}}{\pi^{\frac{n-2}{2}} 2^{\frac{\alpha+n-2}{2}} \Gamma(\frac{\alpha}{2})} J_{\frac{\alpha-n}{2}}(mu^{\frac{1}{2}}) & \text{if } t \in \Gamma_+, \\ 0 & \text{if } t \notin \Gamma_+. \end{cases} \tag{III,1}$$

Here

$$u = t_1^2 + \cdots + t_\mu^2 - t_{\mu+1}^2 - \cdots - t_{\mu+\nu}^2, \qquad \mu + \nu = n; \tag{III,2}$$

$$\mu = 4p + 1, \qquad p = 0, 1, \ldots; \tag{III,3}$$

α is a complex parameter, m a real nonnegative number, n the dimension of the space and $J_\nu(z)$ designates the well-known Bessel function of the first kind defined by the formula

$$J_\nu(z) = \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{z}{2})^{\nu+2p}}{\nu!\,\Gamma(\nu+p+1)}. \tag{III,4}$$

$W_\alpha(u,m)$ which is an ordinary function if Re $\alpha \geq n$, is a distributional function of α. By putting $\mu = 1$ in (III,1) we obtain the kernel introduced by Marcel Riesz (cf. [6], p. 17 and also [7], p. 89).

We put, by definition,

$$\text{(III,5)} \qquad \{\Box + m^2\}^k = \{\frac{\partial^2}{\partial t_1^2} + \cdots + \frac{\partial^2}{\partial t_\mu^2} - \frac{\partial^2}{\partial t_{\mu+1}^2} - \cdots - \frac{\partial^2}{\partial t_{\mu+\nu}^2} + m^2\}^k,$$

$\mu + \nu = n$, m a real nonnegative number and $k = 1, 2, \ldots$.

$\{\Box + m^2\}^k$ is the n-dimensional ultrahyperbolic Klein-Gordon operator, iterated k-times.

We have proved in [5] that $W_\alpha(u,m)$ verifies the following properties:

i) $\{\Box + m^2\} W_{\alpha+2}(u,m) = W_\alpha(u,m)$,
ii) $W_\alpha(u,m) = \sum_{\nu=0}^{\infty} \binom{-\alpha/2}{\nu} m^{2\nu} R_{\alpha+2\nu}(u)$,
iii) $W_{-2k}(u,m) = \{\Box + m^2\}^k \delta$, $k = 1, 2, \ldots$,
iv) $W_0(u,m) = \delta$,
v) $\{\Box + m^2\}^k W_{2k}(u,m) = \delta$ and
vi) $W_\alpha(u, m = 0) = R_\alpha(u)$.

IV. The formula of composition $W_\alpha * W_\beta(u,m) = W_\alpha(u,m)$.

Taking into account the formula (cf. (IV.9) of [5]):

$$\text{(IV,1)} \qquad W_{-2k}(u,m) = \{\Box + m^2\}^k \delta,$$

$k = 0, 1, \ldots$;
and putting $-2k = \beta$, we have

$$\text{(IV,2)} \qquad W_\beta(u,m) = \{\Box + m^2\}^{-\beta/2} \delta.$$

The formula (VII,2) is formal, because the right-hand member is a fractional power of the Klein-Gordon operator; this has been completely justified. (See [5], pp. 16-19).

Therefore, we obtain

$$\text{(IV,3)} \qquad W_\alpha * W_\beta = W_\alpha * \{\Box + m^2\}^{-\beta/2} \delta = \{\Box + m^2\}^{-\beta/2} W_\alpha.$$

Otherwise, we know, (cf. (IV,2) of [5]) that

$$\text{(IV,4)} \qquad \{\Box + m^2\}^k W_\alpha(u,m) = W_{\alpha-2k}(u,m).$$

Putting $k = -\beta/2$, we arrive at $\{\Box + m^2\}^{-\beta/2} W_\alpha(u,m) = W_{\alpha+\beta}(u,m)$. (IV,5) From (IV,3) and (IV,5), we finaly obtain

$$\text{(IV,6)} \qquad W_\alpha * W_\beta(u,m) = W_{\alpha+\beta}(u,m),$$

and our assertion is proved.

References

[1] NOGIN, V.A., *Inversion of Bessel Potentials*, Plenum Publishing Corporation, pp. 997-1000, 1983.

[2] GONZALEZ DOMINGUEZ, A. and TRIONE S.E., *On the Laplace transforms of retarded, Lorentz-invariant functions*, Advances in Mathematics, Vol. 31, No. 1, pp. 51-62, January 1979.

[3] NOZAKI, Y., *On Riemann-Liouville integral of ultrahyperbolic type*, Kodai Mathematical Seminar Reports, Vol. 6, No. 2, pp. 69-87, 1964.

[4] TRIONE S.E., *On Marcel Riesz' ultrahyperbolic kernel*, Trabajos de Matemática, No. 116, Preprint, Instituto Argentino de Matemática, IAM-CONICET, 1987.

[5] TRIONE S.E., *On the elementary retarded, ultrahyperbolic solution of the Klein--Gordon operator, iterated k-times*, Trabajos de Matemática, No. 119, Preprint, Instituto Argentino de Matemática, IAM-CONICET, 1987.

[6] RIESZ M., *L'intégrale de Riemann Liouville et le probléme de Cauchy pour l'equation des ondes*, Comm. Sem. Math. Univ. de Lund 4, 1939.

[7] RIESZ M., *L'integrale de Riemann-Liouville et le probléme de Cauchy*, Acta Math. 81, 1-223, 1949.

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Instituto Argentino de Matematica
Consejo Nacional de Investigaciones Científicas y Técnicas
Buenos Aires, Argentina

Recibido por UMA en el mes de mayo de 1989.