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DIFFUSION AND NON LINEAR POPULATION THEORY

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1. Introduction

Throughout this paper we consider a population evolving in a bounded three dimensional habitat (ocean, rain forest with a height distribution, etcetera).

The function $u(x_1, x_2, x_3, t)$ will denote the population density at time t at the point $x = (x_1, x_2, x_3)$. Our bounded habitat will be denoted by the letter G. Outside G and defined throught the whole space we shall consider the population source per unit of time:

$$f(x_1, x_2, x_3, t)\,. \tag{1.1}$$

On G, the population source per unit of time will be given by the expression:

$$c_1(x)\, u\, [\, 1 - \beta(x)\, u\,] \tag{1.2}$$

$$c_1(x) = 0 \text{ outside } G\,.$$

In the above expression, $\beta(x)$ stands for a bounded and continuous function defined on $\mathbf{R}^3$. The function $c_1(x)$ is assumed to be continuous. Clearly, (1.2) represents a generalization of the logistic growth.

We may assume a predatorial action per unit of time on our population U, on G, given by the expression:

$$-\, c_2(x)\, u^2 \tag{1.3}$$

$$c_2(x) = 0 \text{ outside } G\,.$$

As in the case of $c_1(x)$, $c_2(x)$ is a continuous function.Condition (1.3) would indicate that the predatorial action on our population is negligible for small values of the density u. This simply means that the predators switch to alternate preys when the values of U fall below certain levels. Finally, we may assume migration in our population U, which is the shift of population from areas of large density to areas of lesser density represented by the scaled laplacian:

$$D_{11}\, u \;+\; D_{22}\; u + D_{33}\, u\,. \tag{1.4}$$

In the above expressions D_{ii} denotes the second partial derivative with respect to the variable x_i. Likewise, D_i will denote the partial derivative with respect to x_i and D_t, that with respect to t. The rate of change of the population density u with respect to the time is given by the partial $D_t u$.

The general balance law gives the following equation for the rate of growth of u:

$$\text{(1.5)} \qquad Dt\ u - \Sigma_1^3 D_{ii} u = c_1 u [1 - \beta(x) u] - c_2 u^2 + f.$$

We may assume that the initial population distribution is known to be:

$$\text{(1.6)} \qquad u(x, 0) = g(x).$$

The aim of this paper is to study the existence of weak global solutions to the initial value problem (1.5), (1.6) in certain L^p classes. Likewise, the paper explores the existence of steady state solutions (solutions independent from t) when the source f becomes stable, that is, independent from t.

This problem originated in the diffusion equation that governs the spatial patterning of the spruce budworm as studied in [3].In the case of the Ludwig-Jones-Holling-Aronson-Weinberger equation the term f in (1.5) takes the form:

$$\text{(1.7)} \qquad \text{Constant}.\ u^2(1 + u^2)^{-1}.$$

Instead (1.7), I consider here a simpler version of the predatorial action that is not governed by a "logistic" behavior and, as a trade off, one obtains solutions that are not achievable by the "traveling waves" method.

Another important difference is the fact that unlike the setting in [3], this paper presents a three dimensional set up. The reason for that important dimensional difference is the fact that, some times, it is necesary to describe spatial distributions of population densities not only in their surface dispersal, but also in their height or depth variation.

As indicated before, a third dimension is meaningful when describing oceanic distributions of fish populations whose depth range is wide and constitutes a non negligible dimension of the habitat.

Likewise, in the case of the rain forests, the ecological distribution varies with the height range. Many species cover a wide range of altitudes, and in order to describe their interplay is quite natural to consider three dimensional densities.

This paper focuses on the particular problem: "Suppose that the density distribution of a population U is known throughout $\mathbf{R}^3$ and, at an instant $t = 0$, a new bounded habitat G opens up for the species to migrate in. If we assume a logistic growth for the species U as well as a predatorial action within G, as described in (1.2) and (1.3), find the density distribution of the population U in G for all time $t > 0$, assuming that the migration is governed by diffusion".

As a simplification, we may consider that the population source outside G is given by the function f as in (1.1) and furthermore, we shall assume

$$\text{(1.8)} \qquad f = 0,\ x \in G,\ t > 0.$$

Within G, we shall assume the growth of the density u per unit of time as described in (1.2) and the predatorial action as described in (1.3).

The values of u at $t = 0$ will be given by $u_0(x)$. Obviously, we must have:

(1.9) $$u_0 = 0\,,\, x\ \in G\,,\, t = 0\,.$$

As a simplification, we neglect the description of any natural barrier beyond G, assuming that any diffusion of the biomass toward infinity can be interpreted as a loss due to inhospitable subhabitats. At any rate, the distortion caused by the "diffusion toward infinity" can be compensated by the selection of an appropiate source function f.

Finally, the problem can be set up as in (1.5) and (1.6) by making the appropiate selection of f and g as in (1.8) and (1.9).

Classes Of Functions And Statement Of The Main Result.

E(x,t) will denote the fundamental solution of the heat equation in $\mathbf{R}^3$, namely:

(1.10) $$E\,(x\,,\,t) = (4\,\pi\,t)^{-3/2} \exp - \{\, |x|^2\ (4\,t)^{-1}\,\}$$

$$|\,x\,| = (x_1^{\,2} + x_2^{\,2} + x_3^{\,2})^{1/2}\ .$$

L(u) will denote the Heat differential operator applied to u, (left hand side of equation (1.5)). Hence, the equation (1.5) can be written as:

(1.11) $$L(u) = c_1\, u(1 - \beta u) - c_2\, u^2 + f\,.$$

E(v) will denote the convolution:

(1.12) $$E(v) = \int_0^t \int_{\mathbf{R}^3} E(x\text{-}y,\, t\text{-}s)\ \ v\,(y\,,\,s)\ \ dy\, ds\ .$$

W(g) will denote the convolution on the spatial variables:

(1.13) $$\int_{\mathbf{R}^3} E(x\text{-}y,\, t)\ \ g(y)\ \ dy\ .$$

The equation (1.10) is going to be rewritten as :

(1.14) $$L(u) = -\, a^2\, u^2 + b^3\, u + f$$

$$a^2 = c_1\, b + c_2\ ,\ \ b = (c_1)^{1/3}\,, c_1 \geq 0\,, c_2\ \geq 0\,,\ \beta \geq 0\ .$$

Solving the equation (1.5) with initial data (1.6), if one assumes enough regularity on u, a, b, g and f, is equivalent with solving the integral equation:

(1.15) $$u = E(-\, a^2\, u^2 + b^3\, u + f) + W(g)\ \ .$$

We shall call any solution u for all $t > 0$ of (1.15) a weak global solution of the equation (1.5) with initial data (1.6) whenever the integrals that are involved exist in the Lebesgue sense for all value $t > 0$.

Since the local behavior of solutions of the problem (1.5), (1.6) have no biological meaning, we will consider in our discussion only properties of weak solutions that are global in nature.

$\|\, g\,\|_p$ will denote the usual L^p norm of the function g in $\mathbf{R}^3$.

$\| v^* \|_p$ will denote the L^p norm in $\mathbf{R}^3$ of the function:

$$v^*(x) = \sup_{t>0} |v(x,t)| \ . \tag{1.16}$$

The main result of this paper is contained in the following:

Theorem A.
There exist two small constants $\epsilon_0 > 0$ *and* $\delta_0 > 0$ *such that whenever*

$$\| W(g)^* \|_{9/2} + \| f^* \|_{9/8} < \epsilon_0 \ , \tag{1.17}$$

$$(\| b \|_{9/2})^3 < \delta_0 \ .$$

The problem (1.5) *with initial data* (1.6) *possesses a global weak solution* u(x,t) *that satisfies:*

i) $\| u^* \|_{9/2} < C_0$.

Here, C_0 *depends on* ϵ_0 *and on* δ_0.
Concerning steady state solutions, if the source function f does not depend on t and satisfies

$$\| f \|_{9/8} < \epsilon_1 \tag{1.18}$$

and b *satisfies*

$$(\| b \|_{9/2})^3 < \delta_1 \ .$$

Here ϵ_1 *has the same meaning as* ϵ_0 *above, aldough its numerical value is not necessarily the same. The same for* δ_1.
Then, there exists a steady state solution u that satisfies:

ii) $\| u \|_{9/2} < C_1$.

Here, C_1 *depends on* ϵ_1 .

2. Proof Of Theorem A.

A Potential Inequality.

Lemma 1.
Let $T(v_1, v_2, v_3, v_4)$ *be the multilinear operator defined by:*

$$|x|^{-1} * v_1 \,.\, v_2 \,.\, v_3 \,.\, v_4 \tag{2.1}$$

Here, the functions v_i *are measurable and defined on* $\mathbf{R}^3$, $|x|$ *is the distance from the origen, and* $*$ *is the convolution symbol. Then:*

i) $\| T \|_{9/2} < C \ \| v_1 \|_{9/2} \ \| v_2 \|_{9/2} \ \| v_3 \|_{9/2} \ \| v_4 \|_{9/2}$

Proof.
The Hardy-Littlewood-Sobolev inequality gives, see ref [5] p.119:

(2.2) $\quad \| T \|_q < C_p \| v_1 . v_2 . v_3 . v_4 \|_p \quad 1/q = 1/p - 2/3 .$

Take $p = 9/8$ and apply Hölder's inequality to the right hand side of (2.2) above, $p_i = 9/2$, $i = 1, 2, 3, 4$.

Lemma 2.
Consider the convolution $E(v)$, *where* $v = v(x, t)$ *is a measurable function in* $\mathbf{R}^3 x \mathbf{R}_+$. *If* v^* *belongs to some* L^p *class, we have :*

i) $|E(v)| \leq C_0 |x|^{-1} * v^*$.

Proof.
The above estimate is a consequence of:

(2.3) $\quad |E(x, t)| \leq C(|x| + t^{1/2})^{-3}$.

Taking in the convolution E(v) the integral with respect to the time as the inner integral and using the estimate (1.16) and (2.3), we obtain i) for the particular value of C_0

(2.4) $$C \int_0^\infty (1 + t^{1/2})^{-3} \, dt .$$

This observation concludes the proof.

Estimates for the integral equation.
Calling $F = E(f)$ and observing that as a consequence of Lemma 2 and the Hardy-Littlewood-Sobolev potential inequality we have:

(2.5) $\quad \| F^* \|_{9/2} \leq C \| f^* \|_{9/8}$.

Denoting by $\| \ \|$ the norm $\| ( )^* \|_{9/2}$, we obtain for the operator:

(2.6) $\quad T(u, v) = E(- a^2 u v + b^3 u + f) + W(g)$

the estimate:

(2.7) $\quad \| T(u, v) \| \leq C \{ \| a \|^2 \| u \| \ \| v \| + \| b \|^3 \| u \| + \| F \| + \| W(g) \| \}$

wich is a consequence of lemmae 1 and 2.
On the other hand, the integral equation can be written as:

(2.8) $\quad u = T(u, u)$.

Lemma 3.
Let $T(u,v)$ *be a general operator of the type* (2.6), *mapping the cartesian product* $X \times X$ *into* X, *where* X *denotes a Banach space, such that:*

(2.9) $$\|T(u,v)\| \leq C_1 \|u\| \, \|v\| + C_2 \|u\| + \|F\|$$

Suppose that C_1, C_2 and $\|F\|$ *satisfy:*

(2.10) $$(1 - C_2)^2 > 4 C_1 \|F\| \ , \quad 0 \leq C_2 < 1 \ , \quad C_1 > 0$$

Then, the quadratic operator $T(u, u)$ *maps the ball* $\{\|u\| \leq s_1\}$ *into itself if* s_1 *is the smallest root of the equation:*

(2.11) $$C_1 s^2 + (C_2 - 1) s + \|F\| = 0 \ .$$

If $2 s_1 C_1 + C_2 < 1$, $T(u,u)$ *is a contraction mapping in the ball of radius* s_1. *Finally, we have for* s_1 *the estimates:*

(2.12) $$0 < s_1 < \|F\| \ \{(1 - C_2)^2 - 4 C_1 \|F\|\}^{-1/2} \ .$$

For the proof of this lemma I refer the reader to ref [4] (lemma (2.2) there).

Lemma 3 applied to the (2.8) gives immediately part i) of theorem A.

Part ii) follows by using the same arguments as in i), and it reduces to solving the integral equation:

(2.13) $$u = C\,|x|^{-1} * (- a^2 u^2 + b^3 u + f) \ .$$

Here, the norms we use are the usual $L^{9/2}(\mathbf{R}^3)$ and $L^{9/8}(\mathbf{R}^3)$ norms. The symbol * denotes convolution on the spatial variables. This concludes the proof of Theorem A.

Final remarks.

If the initial density g of the population U as well as the source function remain bellow certain levels, the population in G will not experience an outbreak, even if one supresses the predatorial action.

From the method that we have employed, it follows that the steady state solution is stable, this however will not be done here.

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