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A BEER'S THEOREM IN UNIFORM SPACES

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ABSTRACT. The fact that completeness and total boundedness become a characterization of compactness in the category of uniform spaces is well known. Beer [1] establishes a new characterization for chainable metric spaces based on two conditions: uniform local compactness, which is stronger than completeness, and uniform chainability, weaker than total boundedness. In this paper, we extend this result to the context of all uniform - T_2 spaces.

1. PREVIOUS DEFINITIONS.

Let X be a Tychonoff space endowed with a diagonal uniformity $\mathcal{D}$. We say that a subset A of X has a *lower diameter than* D $\in \mathcal{D}$ if there exists a point x in X such that A $\subset$ D[x]. If p,q are points of X, a D-*chain of lenght* n, *from* p *to* q is a sequence $a_0,\ldots,a_n$ in X, being a_0 = p and a_n = q, such that given a_{j-1},a_j, i = 1,...,n there exists a subset A_j of X with a lower diameter than D containing (a_{j-1},a_j). Let us call $(X,\mathcal{D})$ D-*chainable* if each two points in X can be joined by a D-chain. $(X,\mathcal{D})$ will be called a *chainable* space if it is D-chainable for each D $\in \mathcal{D}$.

Let us suppose $(X,\mathcal{D})$ a chainable uniform space. Then X is *uniformly* D-*chainable* if there is a positive integer n_D such that any pair of points in X can be joined by a D-chain of length

n_D at most. $(X,\mathcal{D})$ is called *uniformly chainable* if it is uniformly D-chainable for each $D \in \mathcal{D}$.

On the other hand, extending the condition of uniform-local compactness given by Beer for metric spaces, we say that a uniform-T_2 space is a *uniformly locally compact* one if there exists a surrounding D such that $\overline{D[x]}$ is compact for all x in X, where $D[x] = \{y \in X / (x,y) \in D\}$. However, for operational reasons we introduce an alternative way: $(X,\mathcal{D})$ is called a *uniformly locally compact space* if and only if X admits a $D \in \mathcal{D}$ such that $\overline{D}[x]$ is compact for each $x \in X$, [2], being $\overline{D}[x] = \{y \in X / (x,y) \in \overline{D}\}$.

2.

LEMMA 1.([2]). *The collections* $\{\overline{D[x]} / D \in \mathcal{D}\}$ *and* $\{\overline{D}[x] / D \in \mathcal{D}\}$ *are equivalent local systems of neighborhoods for the topology of* X.

LEMMA 2. *Let* $(X,\mathcal{D})$ *be a chainable uniform space. If* X *is totally bounded, then* X *is uniformly chainable. If* X *is uniformly locally compact, then* X *is complete.*

Proof. For X totally bounded and $D \in \mathcal{D}$, set $\{U_1,\ldots,U_n\}$ a finite cover of X such that $U_k \times U_k \subset D$, $k = 1,\ldots,n$. It follows that each U_k has a lower diameter than D. If we choose a point a_i in U_i, $i = 1,\ldots,n$, and $\Phi_{D(a_i,a_j)}$ denotes the length of the shortest D-chain joining a_i with a_j, then there exists U_i and U_j such that $x,a_i \in U_i$; $y,a_j \in U_j$. Thus, x and y can be joined by a D-chain of length $2 + \max\{\Phi_D(a_i,a_j); 1 \leq i,j \leq n\}$ at most, for any $x,y \in X$.

For the second assertion, we needn't use the hypothesis of chainability for $(X,\mathcal{D})$. If we pick up a Cauchy-net $(x_\lambda)_{\lambda\in\Lambda}$ in X and a sorrounding D_0 such that each $\overline{D}_0[x]$ is compact, there exists $\lambda_0 \in \Lambda$ large enough such that the net $(x_\lambda)_{\lambda\varepsilon\Lambda}$ is residually in the compact subset $\overline{D}_0[x_{\lambda_0}]$. Then a closure point of $(x_\lambda)_{\lambda\varepsilon\Lambda}$ exists in $\overline{D}_0[x_{\lambda_0}]$, which is also a convergence point.

THEOREM. *Let* $(X,\mathcal{D})$ *be a chainable uniform space. Then* X *is compact if and only if* X *is uniformly locally compact and uniformly chainable.*

Proof. Since any $\overline{D}[x]$ is a closed set in the compact X, then X is uniformly locally compact. However X is totally bounded, and by virtue of lemma 2, X is uniformly chainable.

To show the converse, let X be uniformly locally compact and uniformly chainable and let $D_0 \in \mathcal{D}$ such that $\overline{D}_0[x]$ is compact for all x in X. Choose $E \in \mathcal{D}$, $E \circ E \subset D_0$. Lemma 1 allows us to pick up an open and symmetric sorrounding D satisfying $D \subset \overline{D} \subset E$. [2]

Firstly, let C be a closed subset in X; we will show that $\overline{D}[C] = \bigcup_{x\varepsilon C} \overline{D}[x]$ is also a closed set. Suppose that $(x_\lambda)_{\lambda\varepsilon\Lambda}$ is a net in $\overline{D}[C]$ convergent to a point x; there exists $\lambda_0 \in \Lambda$ such that $(x,x_\lambda) \in D$ for $\lambda \geqslant \lambda_0$. Let c_λ a point in C verifying $x_\lambda \in \overline{D}[c_\lambda]$, for each λ; thus $(x,x_\lambda),(c_\lambda,x_\lambda) \in \overline{D}$ $\forall\ \lambda \geqslant \lambda_0$. But since D is symmetric, likewise is $\overline{D}$, and so $(x,c_\lambda) \in \overline{D}_0$. Then, $(c_\lambda)_{\lambda\varepsilon\Lambda}$ is residually in the compact $\overline{D}_0[x]$, and because C is a closed set, a subnet (c_μ) exists, convergent to a point c in C. Moreover, $(c,x) \in \overline{D}$, because the subnet $(c_\mu,x_\mu) \subset \overline{D}$ converges to (c,x); so $x \in \overline{D}[c]$ and $x \in \overline{D}[C]$.

Secondly let A be a compact subset of X, we assert that $\overline{D}[A]$ is also compact. To this end, let $(x_\lambda)_{\lambda\varepsilon\Lambda}$ be a net in $\overline{D}[A]$. As in the previous case, choose $a_\lambda \in A$ such that $(a_\lambda,x_\lambda) \in \overline{D}$; hence a net $(a_\lambda)_{\lambda\varepsilon\Lambda}$ in A is obtained; as A is compact, there exists a closure point a in A for $(a_\lambda)_{\lambda\varepsilon\Lambda}$. If (a_μ) is a subnet convergent to this point, we have $(a,a_\mu) \in D$ if $\mu \geqslant \mu_0$, being μ_0 large enough. Then, $(a,x_\mu) \in \overline{D}\circ\overline{D} \subset \overline{D}_0$ and (x_μ) is residually in the compact $\overline{D}_0[a]$; therefore (x_μ) as well as (x_λ), has a closure point in $\overline{D}_0[a]$. But $\overline{D}[A]$ is a closed set, and that closure point will be in $\overline{D}[A]$.

Finally, given a point p in X, it follows from the last assertions that $\overline{D}[\{p\}]$ is compact, and so is $\overline{D}^k[\{p\}] = \overline{D}[\overline{D}^{k-1}[\{p\}]]$, $(\forall k \in N)$. As X is uniformly chainable a positive integer n exists such that every point in X can be joined to p by a D-chain of length n, at most, showing that $X = \overline{D}^n[\{p\}]$ is compact. q.e.d.

As a consequence of the fact that chainability is a weaker condition than connectedness, and that compact chainable uniform spaces are connected, [4] we conclude the following: *a chainable uniform space is a continuum if and only if it is uniformly chainable and uniformly locally compact.*

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