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ON "GOOD UNIVERSAL WEIGHTS" IN ERGODIC THEORY

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ABSTRACT. Let $\underline{a} = (a_n)$ be a bounded complex sequence such that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j z^j$ exists for all complex number z in the unit circle. In this paper we prove that if the sequence $\underline{a}(k) = (a_n - a_{n+k})$ is a good universal weight for some natural number k then $\underline{a}$ is a good universal weight. In particular, we extend a certain class of sequences for which the Weighted Pointwise Ergodic Theorem holds.

1. INTRODUCTION.

We denote by N the set of nonnegative integers and by C_1 the set of complex numbers z such that $|z| = 1$.

Let (Ω, M, μ) be a probability space and let A be the group of automorphisms of (Ω, M, μ); $T \in A$ if $T: \Omega \to \Omega$ is a bijection which is bimeasurable and preserves μ. Each $T \in A$ induces an operator U_T on $L^p(\Omega) = L^p(\Omega, M, \mu)$, $1 \leq p < \infty$, defined by $U_T f = f \circ T$.

Now, let T be a continuous linear operator on $L^1(\Omega)$ and let $\underline{a} = (a_n)$ be a sequence of complex numbers.

DEFINITION 1.1. We say that $\underline{a}$ is a *good weight* for T if, for every $f \in L^1(\Omega)$

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \, T^j f(\omega) \text{ exists } \mu\text{-a.e.}$$

In the case when $T \in A$ we say that $\underline{a}$ is a *good weight for* T if $\underline{a}$ is a good weight for the operator U_T induced by T.

DEFINITION 1.2. A bounded complex sequence $\underline{a}$ is said to be a *good universal weight* if $\underline{a}$ is a good weight for every Dunford-Schwartz operator.

It is known that $\underline{a}$ is a good universal weight iff $\underline{a}$ is a good weight for every $T \in A$ (see [1]).

We denote by $\ell(\infty)$ the space of all bounded complex sequences and we write $\|\underline{a}\|_\infty = \sup_n |a_n|$, for $\underline{a} \in \ell(\infty)$. We also say that $\underline{a} = (a_n)$ *has a mean* if $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j$ exists. This last number will be denoted by $m(\underline{a})$.

A.Bellow and V.Losert proved (see [3]) the following result.

THEOREM 1.3. *Let* D *be the set of all* $\underline{a} \in \ell(\infty)$ *satisfying the following conditions:*

(1) $\gamma_a(k) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_{j+k} \cdot \overline{a_j}$ *exists for each* $k \in N$.

(2) *The spectral measure corresponding to* $\underline{a}$ *is discrete.*

(3) *The amplitude* $\Gamma_a(z) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \, \overline{z}^j$ *exists for all* $z \in C_1$.

Then every $\underline{a} \in D$ *is a good universal weight.*

Now, for each natural number k, let U_k be the class of all $\underline{a} \in \ell(\infty)$ such that $\Gamma_a(z)$ exists for all $z \in C_1$ and the sequence $\underline{a}(k) = (a_n - a_{n+k})$ is a good universal weight.

By D_k we mean the class of all $\underline{a} \in U_k$ such that $\underline{a}(k) \in D$.
A direct calculation prove that $D \subset D_1 \subset D_k$, for all k, and in [2] it is shown that D_1 is strictly larger than D.

In this paper we will prove that if $\underline{a} \in U_k$ then $\underline{a}$ is a good universal weight. In particular, every sequence $\underline{a} \in \bigcup_k D_k$ is a good universal weight. From the above considerations, it follows that this result generalizes Theorem 1.3.

2. STATEMENTS AND PROOFS.

We start with the following lemma.

LEMMA 2.1. *Let* q,r *be integer numbers,* $0 \leqslant r < q$, *and let* $\underline{a} = (a_n) \in \ell(\infty)$ *such that* $\Gamma_a(z)$ *exists for all* $z \in C_1$. *Then the sequence* $(a_{j.q+r})_{j\in N}$ *has a mean.*

Proof. Let $z_1, z_2, \ldots, z_q$ be the set of q-th roots of unity. Then

$$\Gamma_a(z_i) = \lim_n \frac{1}{q.n} \sum_{j=0}^{q.n-1} a_j \, \bar{z}_i^j =$$

$$= \lim_n \sum_{s=0}^{q-1} \left(\frac{1}{q.n} \sum_{j=0}^{n-1} a_{j.q+s}\right) \bar{z}_i^s$$

For each integer number m, a straightforward calculation shows that

$$\sum_{i=1}^{q} z_i^m = \begin{cases} q \text{ if } m \text{ is a multiple of } q \\ 0 \text{ otherwise} \end{cases}$$

Thus, we get

$$\sum_{i=1}^{q} z_i^r \, \Gamma_a(z_i) = \lim_n \sum_{s=0}^{q-1} \left(\frac{1}{q.n} \sum_{j=0}^{n-1} a_{j.q+s}\right) . \sum_{i=1}^{q} z_i^{r-s} =$$

$$= \lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_{j.q+r} ,$$

and the lemma is proved.

COROLLARY 2.2. *Let* $\underline{a} = (a_n)$ *be a sequence satisfying the conditions of Lemma 2.1. If* $\underline{b} = (b_n)$ *is a periodic complex sequence then the sequence* a.b $= (a_n . b_n)$ *has a mean.*

Proof. Let $p \in N$ such that $b_{j+p} = b_j$ for all $j \in N$. For each $n \in N$ let $q_n \in N$ satisfying $p.q_n \leq n < p(q_n+1)$.

Thus

$$\frac{1}{n} \sum_{j=0}^{n-1} a_j \, b_j = \frac{1}{n} \sum_{s=0}^{p-1} b_s \sum_{j=0}^{q_n-1} a_{j.p+s} + \frac{1}{n} \sum_{j=p.q_n}^{n} a_j \, b_j \, .$$

Since $\lim_n \frac{q_n}{n} = \frac{1}{p}$, from lemma 2.1 we deduce that

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \, b_j = \frac{1}{p} \sum_{s=0}^{p-1} b_s \, m((a_{j.p+s})_{j\varepsilon N}).$$

We can now state the following theorem.

THEOREM 2.3. *Let* k *be a natural number. Then every sequence* $\underline{a} \in U_k$ *is a good universal weight.*

Proof. Let $T \in A$ and let $\underline{a} = (a_n) \in U_k$. We write

$$A_n f(\omega) = \frac{1}{n} \sum_{j=0}^{n-1} a_j \, f(T^j\omega) \quad , \quad f \in L^1(\Omega).$$

Let us consider the set of all functions h which can be represented in the form

$$h(\omega) = g(\omega) - g(T^{-k}\omega) \ ,$$

where g is a bounded function. For any function h as above, we have

$$A_n h(\omega) = \frac{1}{n} \sum_{j=0}^{n-1} (a_j - a_{j+k}) \, g(T^j\omega) + R_n g(\omega) \ ,$$

being $\quad |R_n g(\omega)| \leq \dfrac{2k \, \|\underline{a}\|_\infty \, \|g\|_{L^\infty(\Omega)}}{n}$.

Since the sequence $\underline{a}(k) = (a_j - a_{j+k})$ is a good universal weight, we see at once that $A_n h(\omega)$ converges for almost all ω as $n \to \infty$.

Now, we consider the set of all functions $p \in L^2(\Omega)$ satisfying $p(\omega) = p(T^k\omega)$ μ-a.e.. For any such a function p we can find a set $\Omega_p \subset \Omega$ of full measure such that the sequence $(p(T^j\omega))$ is k-periodic for any $\omega \in \Omega_p$ (by k-periodic we mean that $p(T^{j+k}\omega) = p(T^j\omega)$ for all $j \in N$). By corollary 2.2, $A_n p(\omega)$ converges for every $\omega \in \Omega_p$.

We conclude that $A_n f(\omega)$ converges almost everywhere if f is in the linear span V of the functions h and p. Theorem 2.3 will follow by a standard argument if we prove that V is dense in $L^1(\Omega)$. For this purpose, we assume that for a certain function $q_0 \in L^2(\Omega)$ we have

$$\int_\Omega q_0(\omega)\ \overline{f}(\omega)d\mu = 0 \qquad \text{for all} \quad f \in V.$$

Hence

$$0 = \int_\Omega q_0(\omega)\ \overline{h}(\omega)d\mu = \int_\Omega q_0(\omega)(\overline{g}(\omega)-\overline{g}(T^{-k}\omega))d\mu =$$

$$= \int_\Omega \overline{g}(\omega)(q_0(\omega)-q_0(T^k\omega))d\mu \ ,$$

for every bounded function g.

Then $q_0(\omega) = q_0(T^k_\omega)$ for almost all ω and so $q_0 \in V$. Consequently, we have $\int_\Omega q_0(\omega).\overline{q}_0(\omega)d\mu = 0$, which proves that V is dense in $L^2(\Omega)$. Since $L^2(\Omega)$ is dense in $L^1(\Omega)$, the result follows.

REMARK. A bounded complex sequence $\underline{a}$ such that $\underline{a}(k)$ is a good universal weight does not necessarily have amplitude $\Gamma_a(z)$ for every $z \in C_1$. The following is an example:

Let $k \in N$ and let z_0 be a root of unity of order k. For each $m \in N$ let I_m be the integer interval

$$I_m = \{n \in N/2^m \leqslant n < 2^{m+1}\}.$$

Let α and β be two real and nonnegative numbers. We define the sequence $\underline{a} = (a_n)$ in the following way:

$$a_n = \begin{cases} \alpha \bar{z}_0^n & \text{if } n \in I_m, \text{ m even} \\ \beta \bar{z}_0^n & \text{if } n \in I_m, \text{ m odd.} \end{cases}$$

We see that $a_{n+k} - a_n = 0$ if n and n+k are in I_m, for any m. Then, $\{n \in N/a_{n+k} - a_n \neq 0\}$ has zero density. From this we immediately deduce that $\underline{a}(k)$ is a "good universal weight".

On the other hand,

$$a_n z_0^n = \begin{cases} \alpha & \text{if } n \in I_m, \text{ m even} \\ \beta & \text{if } n \in I_m, \text{ m odd} \end{cases};$$

and a simple calculus shows that $\underline{a}$ has not amplitude in z_0.

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