**Revista de la**
**Unión Matemática Argentina**
**Volumen 36, 1990.**



# A NOTE ON LAGRANGE INTERPOLATION IN $R^2$

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ABSTRACT. In [1] Chung and Yao introduced some sets in $R^2$, satisfying what they called the *GC condition* for $P_n$ unisolvency of Lagrange interpolation problems; in [2] Gasca and Maeztu, while studying which of their *reversible systems* for Hermite interpolation in $R^2$ were $P_n$ unisolvent, conjectured that in the Lagrange case these systems included as a special case those satisfying the *GC condition*, and they mentioned a proof for n=3. In this work we shall prove this conjecture for n=4.

## INTRODUCTION.

By $P_n$ we denote the real polynomials over $R^2$ with total degree less than or equal to n; $P_n$ is a (n+1)(n+2)/2 dimensional linear space. $A \subset R^2$ is said $P_n$ unisolvent when for any set $\{f_a: a \in A\}$ of real numbers, there is one and only one polynomial p in $P_n$ such that $p(a) = f_a$ for all a in A. Given a $P_n$ unisolvent set A, we call *basis function* associated to a point a in A to the function $p_a$ in $P_n$ such that $p_a(a) = 1$ and $p_a(b) = 0$ for all b in A different from a. In this case, any polynomial p in $P_n$ may be expressed by a *Lagrange formula* as $p = \Sigma_a p(a) p_a$.

## THE GC CONDITION.

Let $A \subset R^2$ be a set with $(n+1)(n+2)/2$ points. We say that A satisfies the *GC condition* for $P_n$ unisolvency when for each a in A there is a set $L_a$ of n lines such that: i) for each $L \in L_a$, $a \notin L$, and ii) if $b \in A$ and $b \neq a$, then there exists $L \in L_a$ such that $b \in L$. When $L \in L_a$ we say that a *needs* L, and we call the points in A *nodes*.

Chung and Yao proved in [1] that if A satisfies the GC condition for $P_n$ unisolvency, then A is $P_n$ unisolvent; in fact, the basis function associated to a node a is easily obtained as the product of the n linear factors $w_L$, where $w_L(x) = 0$ is a linear equation for $L \in L_a$ and the $w_L$ are chosen such that $w_L(a) = 1$.

## GASCA AND MAEZTU'S CONJECTURE.

Let $A \subset R^2$ be a set with $(n+1)(n+2)/2$ points and let it satisfy the GC condition for $P_n$ unisolvency. Then it is easy to see that no line passes through more than $(n+1)$ nodes; Gasca and Maeztu conjectured that *there is at least one line passing through* $(n+1)$ *nodes* (see [2], pp.9-10). We shall not refer to the significance of this conjecture in the context of their work, but we shall prove it for n=4 (for n=1 and n=2, the result is immediate; for n=3 the proof is comparatively easy, as mentioned in [2]).

## THE CASE n=4 OF THE CONJECTURE.

We shall need the following lemma, which is a consequence of a well known result on cubics, which is itself a special case of the Cayley-Bacharach theorem (see [3], pp.671-673); this last theorem seems to be very related to the conjecture in its general version.

LEMMA 1. *Suppose that we have three lines* $L_i$, $i=1,2,3$, *and another three lines* $L'_i$, *such that for each* $i,j$ $L_i$ *and* $L'_j$ *intersect at a point* $a_{ij}$, *and assume that these intersection points* *are all different. Then if a polynomial* $p \in P_3$ *vanishes at eight of the* $a_{ij}$*, it vanishes at the nine.*

Assume from now on that $A \subset R^2$ is a set with fifteen points, and that it satisfies the GC condition for $P_4$ unisolvency.

LEMMA 2. *Suppose that a node* a *needs a line* L *that passes through exactly four nodes; then there are three nodes that need a same line.*

*Proof.* Consider the line L and the four lines that join the node a with the nodes on L: each of the eleven nodes that are not on L needs some of these five lines, thus at least three of them need a same one.

LEMMA 3. *Let two nodes* a *and* b *be such that* $L_a$ *and* $L_b$ *have exactly one line* L *in common. Then* L *has at least four nodes, and if a third node* c *needs* L*,* L *has five nodes.*

*Proof.* The three remaining lines in $L_a$ intersect the three remaining lines in $L_b$ at a set B having at most nine nodes; B must include all the nodes but a,b and those on L, so that L has at least four nodes. Assume now that L passes through exactly four nodes: then B has nine nodes and no node in common with L; a third node c that needs L must be in B, but then the three lines in $L_c$ different from L should cover eight points of B and not the nine, which is impossible by Lemma 1.

LEMMA 4. *If two nodes* a *and* b *are such that* $L_a$ *and* $L_b$ *have exactly two lines in common, then some of these lines passes through five nodes.*

*Proof.* The remaining two lines in $L_a$ intersect the remaining two lines in $L_b$ at a set B that has at most four nodes. B must contain all the nodes but a, b and those on some of the lines common to $L_a$ and $L_b$; thus on these two lines there are at least nine nodes, so that some of them passes through five nodes.

LEMMA 5. *Suppose that three nodes* a,b *and* c *need a same line, and assume that no line in* $L_a$ *passes through five nodes. Then we have:* i) *There are three lines* $L_1$, $L_2$ *and* $L_3$ *needed by both* a *and* b*;* ii) *each of the lines* $L_i$ *passes through exactly four nodes, and two of them have no node in common;* iii) *if a node* d *needs some of the lines* $L_i$*, it needs the three;* iv) *the node* c *also needs the three lines* $L_i$.

*Proof*. i) By Lemma 3, $L_a$ and $L_b$ have more than one line in common; by Lemma 4 they have more than two lines in common; thus $L_a$ and $L_b$ have three lines in common.
ii) As no line in $L_a$ passes through five nodes, the lines $L_i$ cover at most twelve nodes. Let d be a node not covered by them and different from a and b: then the fourth line in $L_a$ must be the one through b and d, and the fourth line needed by b must be the one through a and d; as these lines intersect only at d, there is not a fourth node out of the $L_i$'s so that these lines cover exactly twelve nodes, each of them passes through four nodes, and each two of them have no node in common.
iii) Suppose that d needs $L_1$ and d is not on $L_2$. Then the three remaining lines in $L_d$ cover the four nodes in $L_2$, so that d needs $L_2$; now the two ramaining lines in $L_d$ cover at least three nodes on $L_3$ (all but eventually d), so that d needs $L_3$.
iv) As c needs some of the $L_i$'s, iv) is a consequence of iii).

REMARK. Note that we have thus far shown that if there are no lines through five nodes, then if a line is needed by two nodes it passes through four nodes (Lemmas 3,4 and 5).

THEOREM. *There is a line passing through five nodes.*

*Proof*. Suppose that there is no line passing through five nodes. Then for each node x there is a line in $L_x$ that passes exactly through four nodes, so that by Lemma 2 there are three nodes a,b and c that need a same line; now by Lemma 5 a,b and c need three lines $L_1$, $L_2$ and $L_3$, each of them passing through exactly four nodes and each two of them with no node in common. Let us call B the set of nodes covered by $L_1$, $L_2$ and $L_3$, and C the set of nodes covered by $L_1$ and $L_2$; thus B has twelve nodes and C has eight nodes.

*If three nodes in* B *need a same line, they are each on a different* $L_i$: because (by Lemma 5) they must need three lines in common, each with four nodes and with no node in common between them, so that (as no of these lines can be one of the $L_i$'s) each of these lines have one node on each $L_i$, thus they cover three nodes on each $L_i$ leaving only one node not covered.

*If two nodes in* C *need a same line* L, *it has one node on each* $L_i$: as a,b and c are not on a same line, L has at most two of these nodes; by our previous remark L has four nodes, so that L has one node on at least two of the $L_i$'s, say $L_2$ and $L_3$; let us call d the node on L and $L_2$, and consider the set V formed up with the three lines joining d with the nodes in $L_3$ that are not on L. If a node in C different from d does not need L, it needs some line in V, and then it follows (as there are not three nodes in C needing a same line) that there are at least two lines in V needed by two nodes in C and which in consequence have four nodes. We conclude that L has at most one of the nodes a,b and c, thus it has a node on each $L_i$.

Consider now the set W of lines that join a to the nodes on $L_3$: each of the eight nodes in C needs some of the lines in W, and no three of them need a same one: thus, each line in W is needed by exactly two nodes in C, and in consequence each line in W has a node on each $L_i$. Thus, the four lines of W cover B.

Consider now the polynomial p obtained as the product of the four linear factors $w_L$, where $w_L(x) = 0$ is a linear equation for $L \in W$: as this fourth degree polynomial vanishes at all the nodes but b and c, it should be a linear combination of the basis functions associated to these nodes; but these basis functions vanish all along the lines $L_i$, so that this should be true also for p, which is a contradiction.

FINAL REMARKS. Of course our purpose while studying this special case of the conjecture was to get a thorough understanding of the general one; but we arrived to solve this case with arguments that do not seem to have a wider application; perhaps the most promising in this sense is the relation with the Cayley-Bacharach theorem that we have already mentioned.

## REFERENCES


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Recibido en junio de 1988.