Revista de la
Unión Matemática Argentina
Volumen 36, 1990.

MATHER'S POLYNOMIAL DIVISION THEOREM AND DIVIDED DIFFERENCES

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1. INTRODUCTION.

The aim of this paper is to give a proof of Mather's Polynomial Division Theorem (PDT) based on elementary concepts of Divided Differences and of Glaeser's Composite Mapping Theorem. In fact, PDT is an almost easy corollary of the C^∞-Newton Theorem ([3]). The proof of Glaeser's theorem is, however, somewhat formidable. So, we give here a slightly more elementary proof of this theorem under more restricted conditions though sufficient in our context.

We feel that it is very natural to view PDT as an interpolation problem of Hermite type in the following sense: if f is in $C^\infty(\mathbf{R}^n)$ and $t_1,\dots,t_k$ are points of $\mathbf{R}$, we can write

$$f(x)= p_f\{t_1,\dots,t_k\}(x) + (x-t_1)\dots(x-t_k)\,[t_1,\dots,t_k,x]f$$

where $p_f\{t_1,\dots,t_k\}$ is the Hermite polynomial interpolating f at $t_1,\dots,t_k$ and $[t_1,\dots,t_k,x]\,f$ is the k'th divided difference of f at $t_1,\dots,t_k,x$. This equality looks like a polynomial division formula and we can make appeal to a Glaeser type result to see that this is really the case.

This approach could be also carried over to a more general set of interpolating functions other than polynomials (the ECT-systems [5], pag.363). We do not know, however, if this kind of results could be of any use.

2. SYMMETRIC POWERS.

Let $k \geqslant 2$, and consider the action of the symmetric group of permutations S_k over $\mathbf{C}^k$ by permutation of coordinates on $\mathbf{C}^k$. Let $\sigma_1,\dots,\sigma_k$ be the elementary symmetric polynomials in the coordinates $z_1,\dots,z_k$ of $\mathbf{C}^k$. The σ_i are given by the polynomial identity

$$(y+z_1)\dots(y+z_k) = y^k + \sigma_1(z)y^{k-1} + \dots + \sigma_k(z) \quad , \; y \in \mathbf{C}.$$

We define a map $\pi_k: \mathbf{C}^k \to \mathbf{C}^k$ by $\pi_k = (\sigma_1,\dots,\sigma_k)$. Hence, π_k is a finite morphism in the complex sense and, because $\mathbf{C}$ is algebraically closed, $\pi_k(\mathbf{C}^k) = \mathbf{C}^k$ ([6]).

Now, we look at $\mathbf{C}^k \cong \mathbf{R}^k$ with its natural real structure, and let d be any positive integer. We consider on $\mathbf{C}^k \times \mathbf{R}^d$ the action of S_k by permutation of complex coordinates of $\mathbf{C}^k$, and we denote as $C^\infty(\mathbf{C}^k \times \mathbf{R}^d)^{S_k}$ the closed subalgebra of the Fréchet algebra $C^\infty(\mathbf{C}^k \times \mathbf{R}^d)$ of S_k-invariant real-valued functions over $\mathbf{C}^k \times \mathbf{R}^d$ (of course, these algebras are considered endowed with the C^∞-Whitney topology).

If $\tau_{k.d}: \mathbf{C}^k \times \mathbf{R}^d \to \mathbf{C}^k \times \mathbf{R}^d$ is the map defined by

$$\tau_{k.d}(z,x) = (\pi_k(z),x) \quad , \quad (z,x) \in \mathbf{C}^k \times \mathbf{R}^d$$

the associated homomorphism

$$(\tau_{k.d})^*: C^\infty(\mathbf{C}^k \times \mathbf{R}^d) \to C^\infty(\mathbf{C}^k \times \mathbf{R}^d)^{S_k}$$

of Fréchet algebras is defined by $(\tau_{k.d})^*(f) = f\circ\tau_{k.d}$.

A key step for proving PDT is:

THEOREM 2.1. *For any positive integers* k *and* d, $(\tau_{k.d})^*$ *is an isomorphism of Fréchet algebras.*

Before proving this theorem we turn to explain how it applies to PDT, after a glimpse at interpolation theory.

3. DIVIDED DIFFERENCES. ([2],[5]).

Let $\mathcal{U}_m = \{u_i\}_1^m$ be a set of functions on **R** and let $t_1, t_2, \ldots, t_m$ be points in **R** such that

$$t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m.$$

Then, we define the matrix associated with $\{u_i\}_1^m$ and $\{t_i\}_1^m$ by

$$M\begin{pmatrix} t_1, \ldots, t_m \\ u_1, \ldots, u_m \end{pmatrix} = \begin{pmatrix} u_1(t_1) & u_2(t_1) & \ldots & u_m(t_1) \\ \vdots & \vdots & & \vdots \\ u_1(t_m) & u_1(t_m) & \ldots & u_m(t_m) \end{pmatrix} \tag{3.1}$$

and the determinant

$$D\begin{pmatrix} t_1, \ldots, t_m \\ u_1, \ldots, u_m \end{pmatrix} = \det M\begin{pmatrix} t_1, \ldots, t_m \\ u_1, \ldots, u_m \end{pmatrix} \tag{3.2}$$

Such matrices arise in the basic interpolation problems of Lagrange, Hermite, etc. ([5], pag.20).

Let $\mathcal{U}_m = \{1, x, \ldots, x^{m-1}\}$. Then

$$V(t_1, \ldots, t_m) = D\begin{pmatrix} t_1, \ldots, t_m \\ 1, \ldots, x^{m-1} \end{pmatrix}$$

is the *Vandermonde* determinant.

DEFINITION 3.1. Given a function f and points $t_1, \ldots, t_{r+1}$, $(r \geqslant 0)$, we define the r'th order divided difference over the points $t_1, \ldots, t_{r+1}$ by

$$d_r f(t_1, \ldots, t_{r+1}) = \frac{D\begin{pmatrix} \bar{t}_1, \ldots\ldots, \bar{t}_{r+1} \\ 1, x, \ldots, x^{r-1}, f \end{pmatrix}}{V(\bar{t}_1, \ldots, \bar{t}_{r+1})} \tag{3.3}$$

where $\bar{t}_1 \leq \bar{t}_2 \leq \ldots \leq \bar{t}_{r+1}$ consists of the points $\{t_i\}_1^{r+1}$ in their natural order.

When the t's are distinct, then $d_r f(t_1,\ldots,t_{r+1})$ is defined for any function that has finite values at these points. When one of the t's occurs more than once, then the value of $d_r f(t_1,\ldots,t_{r+1})$ depends on certain derivatives of f.

It is clear from (3.3) that a divided difference is a linear operator on f. In the next theorem we give several important properties of divided differences (see [5]).

THEOREM 3.2. *Given points* $t_1,\ldots,t_{r+1}$ *and any function* f *on* $\mathbf{R}$, *we have:*

If $t_1 = t_2 = \ldots = t_{r+1}$, *then*

$$r!.d_r f(t_1,\ldots,t_{r+1}) = D^r f(t_1) \tag{3.4}$$

In general, if $a = \min\{t_i\}$ *and* $b = \max\{t_i\}$,

$$r!.d_r f(t_1,\ldots,t_{r+1}) = D^r f(\nu) \tag{3.5}$$

for some $a \leq \nu \leq b$.

For $i = 1,\ldots,r+1$, $(\partial/\partial t_i)d_r f$ *exists and, if* $t = (t_1,\ldots,t_{r+1})$

$$\frac{\partial}{\partial t_i} d_r f(t) = d_{r+1} f(\hat{t}_i) \tag{3.6}$$

where $\hat{t}_i = (t_1,\ldots,t_{i-1},t_i,t_i,t_{i+1},\ldots,t_{r+1})$.

COROLLARY 3.3. For any function f on $\mathbf{R}$ and $r \geq 0$, $d_r f$ is a smooth function symmetric in its arguments. Also,

$$d_r : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}^{r+1})$$

is a Frêchet homomorphism (C^∞-Whitney topology).

THEOREM 3.4. Hermite Interpolation. *Given a function* f *and points*

$$t_1 \leq \ldots \leq t_m = \underbrace{\tau_1,\ldots,\tau_1}_{l_1} < \underbrace{\tau_2,\ldots,\tau_2}_{l_2} < \ldots < \underbrace{\tau_d,\ldots,\tau_d}_{l_d}$$

there exists a unique polynomial $p_f\{t_1,\ldots,t_m\}$ *in* P_m = *set of polynomials of degree* $\leq m$, *such that*

$$D^{j-1}p_f\{t_1,\ldots,t_m\}(\tau_i) = D^{j-1}f(\tau_i) \ , \ j = 1,\ldots,l_i \ , \ i = 1,\ldots,d.$$

REMARK 3.5. Using divided differences it is possible to give an exact expression for $p_f\{t_1,\ldots,t_m\}$ and for the difference between f and $p_i\{t_1,\ldots,t_m\}$. Setting $t = (t_1,\ldots,t_m)$, we have:

$$p_f\{t\}(x) = f(t_1) + d_1f(t_1,t_2).(x-t_1) + \ldots + d_{m-1}f(t).(x-t_1). \ \ldots \ .(x-t_{m-1})$$

and

$$f(x) = p_f\{t\}(x) + (x-t_1). \ \ldots \ .(x-t_m).d_mf(x,t)$$

If $p_f\{t\}(x) = \sum_{k=0}^{m-1} c_f^k(t).x^k$, we see that each c_f^k is a smooth function on $\mathbf{R}^n$ and, because $p_f\{t\}$ is univocally determined by the points $t_1,\ldots,t_m$ and not by their order, symmetric in its variables.

4. MATHER'S GLOBAL DIVISION THEOREM.

For every positive integer m, let Γ_m: $\mathbf{R}\times\mathbf{R}^m \to \mathbf{R}$ be the monic polynomial $\Gamma_m(x,\mathbf{u}) = x^m + \sum_{k=0}^{m-1} u_k.x^{m-k}$, $x \in \mathbf{R}$ and $\mathbf{u} \in \mathbf{R}^m$.

We shall reformulate Mather's theorem in such a way that we also have uniqueness of division if we required an additional condition.

THEOREM 4.1. *There exists a unique Fréchet homomorphism*

$$c_i: C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}^m) \ , \ i = 0,\ldots,m-1$$
$$R: C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}\times\mathbf{R}^m)$$

such that:

1) For each $f \in C^{\infty}(\mathbf{R})$ *we have*

$$f(x) = \sum_{k=0}^{m-1} c_k(f)(\mathbf{u}).x^k + \Gamma_m(x,\mathbf{u}).R(f)(x,\mathbf{u})$$

for every $(x,\mathbf{u}) \in \mathbf{R}\times\mathbf{R}^m$.

2) For each $\mathbf{u} \in \mathbf{R}^m$, *let* $\{r_k(\mathbf{u})\}_{k=1}^{m}$ *be the set of roots of* $\Gamma_m(-,\mathbf{u})$ *(of course,* $r_k(\mathbf{u})$ *can be a complex number). Set, for* $k = 1,\dots,m$, $t_k(\mathbf{u})$ *equal to the real part of* $r_k(\mathbf{u})$, *i. e.,* $t_k = \mathrm{Re}(r_k(\mathbf{u}))$.

Then

$$\sum_{k=0}^{m-1} c_k(f).x^k$$

is the Hermite interpolating polynomial of f *at* $\{t_k\}_{k=1}^{m}$

Proof. Uniqueness is straightforward, from 4.1.2.

Existence. For any $\mathbf{z} \in \mathbf{C}^m$ and $k = 1,\dots,m$ we set $t_k = \mathrm{Re}(\mathbf{z}_k)$ and $t(\mathbf{z}) = (t_1(\mathbf{z}),\dots,t_m(\mathbf{z}))$.

If $f \in C^{\infty}(\mathbf{R})$, we have

$$f(x) = p_f\{t(\mathbf{z})\}(x) + \prod_{k=1}^{m}(x-t_k(\mathbf{z})).d_m f(x,t(\mathbf{z}))$$

for every $(x,\mathbf{z}) \in \mathbf{C}^m\times\mathbf{R}$.

The coefficients $\overline{c}_i(f)$, $i = 1,\dots,m$, of $p_f\{-\}$ are smooth real-valued functions over $\mathbf{C}^m$ and symmetric in their arguments, i.e.:

$$\overline{c}_i(f) \in C^{\infty}(\mathbf{C}^m)^{S_m} \quad , \quad i = 0,\dots,m-1$$

and we also have $\overline{R}(f) = d_m f \in C^{\infty}(\mathbf{C}^m\times\mathbf{R})^{S_m}$.

Using theorem 3.2 one can be easily convinced that

$$\overline{c}_i\colon C^{\infty}(\mathbf{R}) \to C^{\infty}(\mathbf{C}^m)^{S_m} \quad , \quad i = 0,\dots,m-1$$

and

$$\overline{R}\colon C^{\infty}(\mathbf{R}) \to C^{\infty}(\mathbf{C}^m\times\mathbf{R})^{S_m}$$

are Fréchet homomorphisms.

Let $i\colon \mathbf{R}^m \to \mathbf{C}^m$ be the natural inclusion and

$$i^*\colon C^{\infty}(\mathbf{C}^m) \to C^{\infty}(\mathbf{R}^m)$$

the associated Fréchet homomorphism. We define now the Fréchet homomorphisms

$$c_i : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}^m) \quad , \quad i = 0,\dots m-1$$

and

$$R' : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}^m \times \mathbf{R})$$

by

$$c_i = i^* \circ [(\tau_{m.0})^*]^{-1} \circ c_i$$

and

$$R' = i^* \circ [(\tau_{m.1})^*]^{-1} \circ \overline{R}.$$

By the construction, it is straightforward to see that the c_i's satisfy 4.1.2. We only need now a slight modification of R' in order to obtain 4.1.1. To accomplish this task we observe that the coefficients $\gamma_k(\mathbf{z})$, k = 0,...,m-1, of the polynomial

$$\prod_{k=1}^{m} (x - t_k(\mathbf{z})) = x^m + \sum_{k=0}^{m-1} \gamma_k(\mathbf{z}).x^k$$

are in $C^\infty(\mathbf{C}^m)^{S_m}$. Hence, there exist functions $\overline{\gamma}_k \in C^\infty(\mathbf{C}^m)$ such that $\gamma_k = \overline{\gamma}_k \circ \pi_m$, k = 0,...m-1. We know then that, for every $\mathbf{z} \in \mathbf{C}^m$ and $\mathbf{w} = \pi_m(\mathbf{z})$, the roots of

$$x^m + \sum_{k=0}^{m-1} \overline{\gamma}_k(\mathbf{w}).x^k$$

are $\{t_k(\mathbf{z})\}_{k=1}^{m}$.

Now, let $\lambda_k \in C^\infty(\mathbf{R}^m)$ be the restriction to $\mathbf{R}^m \subset \mathbf{C}^m$ of $\overline{\gamma}_k$ and P_m the polynomial

$$P_m(x,\mathbf{u}) = x^m + \sum_{k=0}^{m-1} \lambda_k(\mathbf{u}).x^k \ , \ (x,\mathbf{u}) \in \mathbf{R} \times \mathbf{R}^m.$$

By the remark above, $P_m(x,\mathbf{u}) = 0$ when $\Gamma_m(x,\mathbf{u}) = 0$. Therefore, because $\mathrm{grad}(\Gamma_m)$ is never zero on $\mathbf{R} \times \mathbf{R}^m$, by a coordinate change and Taylor's Theorem, there exists $g \in C^\infty(\mathbf{R} \times \mathbf{R}^m)$ such that $P_m = \Gamma_m.g$.

Clearly, the Fréchet homomorphism

$$R : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R} \times \mathbf{R}^m)$$

defined by $R(f) = g.R'(f)$, $f \in C^\infty(\mathbf{R})$, is all we need to finish the proof of theorem 4.1.

5. PROOF OF THEOREM 2.1.

Let $f = (f_1,\dots,f_k)\colon \mathbf{R}^k \to \mathbf{R}^k$ be a surjective and proper polynomial map (f must be open), and $f^*\colon C^\infty(\mathbf{R}^k) \to C^\infty(\mathbf{R}^k)$ the associated Fréchet homomorphism. Clearly, f^* must be injective. We will prove the following Glaeser type theorem:

Claim 5.1. f^* is an isomorphism onto its image. Or, what is the same, $\mathrm{Im}(f^*)$ is closed in $C^\infty(\mathbf{R}^k)$.

By "$g_n \to f$", $\{g_n\}_{n=1}^\infty$ and f in $C^\infty(\mathbf{R}^k)$, we mean that the limit of the sequence of functions g_n is f in the C^∞-Whitney topology. More precisely, for every integer $q \geqslant 0$ and compact set $K \subset \mathbf{R}^k$

$$\lim_{n\to\infty} (\|g_n - f\|_{k,q} = \max\{|D^\beta(g_n - f)(\mathbf{x})|,\ \mathbf{x} \in K,\ |\beta| \leqslant q\}) = 0.$$

Functional Analysis implies that Claim 5.1 is equivalent to:

5.1'. If $\{g_n\}_{n=1}^\infty$ is a sequence in $C^\infty(\mathbf{R}^k)$ such that

$$g_n \circ f = f^*(g_n) \to 0\ ,$$

then $g_n \to 0$.

Let $\mathbf{v}$ be a point in $\mathbf{R}^k$. $C_{\mathbf{v}}^\infty(\mathbf{R}^k)$ is the ring of all germs of smooth real-valued functions defined in a nbhd of $\mathbf{v}$, $m_{\mathbf{v}}$ the unique maximal ideal in $C_{\mathbf{v}}^\infty(\mathbf{R}^k)$, and $f_{\mathbf{v}}$ the germ at $\mathbf{v}$ of f. If $\mathbf{x} \in \mathbf{R}^k$ and $\mathbf{y} = f(\mathbf{x})$ then, moving at germ level, we get morphisms

$$f_{\mathbf{v}}^*\colon C_{\mathbf{y}}^\infty(\mathbf{R}^k) \to C_{\mathbf{x}}^\infty(\mathbf{R}^k)$$

and, for each integer $q \geqslant 0$,

$$(f_{\mathbf{x}}^*)_q\colon C_{\mathbf{y}}^\infty(\mathbf{R}^k)/m_{\mathbf{v}}^{q+1} \to C_{\mathbf{x}}^\infty(\mathbf{R}^k)/m_{\mathbf{x}}^{q+1}\ .$$

$C_{\mathbf{v}}^\infty(\mathbf{R}^k)/m_{\mathbf{v}}^{q+1}$ is naturally isomorphic to the vector space $J^q(k)$ of q-jets of functions at $\mathbf{v} \in \mathbf{R}^k$. Also, if $g \in C_{\mathbf{v}}^\infty(\mathbf{R}^k)$, then

$$(f_v^*)_q(j^q g(y)) = j^q(g\circ f)(x).$$

For every $q \geq n$, there is a canonical projection

$$\pi_{q.n} : J^q(k) \to J^n(k)$$

and we can view $J^n(k)$ as naturally immersed in $J^q(k)$ via the decomposition $J^q = J^n \oplus \mathrm{Ker}(\pi_{q.n})$.

Let $s = \max_{1\leq i\leq k} \{\mathrm{degree}(f_i)\}$. Since $f^*: P(\mathbf{R}^k) \to P(\mathbf{R}^k)$ must be injective ($P(\mathbf{R}^k)$ = ring of polynomials over $\mathbf{R}^k$), the following remark is straightforward.

Claim 5.2. For every $r > 0$ and $\mathbf{x} \in \mathbf{R}^k$

$$(f_x^*)_{sr}: J^{sr}(k) \to J^{sr}(k)$$

is injective over $J^r(k) \subset J^{sr}(k)$. More precisely, if $w_{y.1},\dots,w_{y.k}$ are generators of m_v ($\mathbf{y} = f(\mathbf{x})$), then the vectors

$$\{(f_x^*)_{sr}(w_y^{\alpha}) : |\alpha| \leq r \text{ and } w_y^{\alpha} = \prod_{n=1}^{k} w_{y.n}^{\alpha_n}\}$$

are linearly independent.

Now, let $\{g_n\}_{n=1}^{\infty}$ be a sequence in $C^{\infty}(\mathbf{R}^k)$ such that $g_n\circ f \to 0$. We must show that $g_n \to 0$, that is, for each $r \geq 0$, $g_n \to 0$ in $C^r(\mathbf{R}^k)$.

Because f is proper, open and surjective, it is enough to show claim 5.1 in a local context, that is, it suffices to prove:

Claim 5.3. For every $\mathbf{x}_0 \in \mathbf{R}^k$ there exists a nbhd U of $\mathbf{x}_0$, $\overline{U}$ compact in $\mathbf{R}^k$, such that

$$\lim_{n\to\infty} \|(g_n\circ f)\|_{f(\overline{U}),sr} = 0$$

implies

$$\lim_{n\to\infty} \|g_n\|_{\overline{U},r} = 0$$

Proof. Let F be a vector subspace of $J^{sr}(k)$ which satisfies

$$J^{sr}(k) = (f^*_{x_0})_{sr}(J^r(k) \oplus F$$

and pr: $J^{sr}(k) \to J^{sr}(k)/F = E$ the canonical projection.

Writing $l: \mathbf{R}^k \to L(J^{sr}(k),E)$ to denote the continuous map defined by

$$l(\mathbf{x}) = pr \circ (f^*_{\mathbf{x}})_{sr} \quad , \quad \mathbf{x} \in \mathbf{R}^k$$

we can choose a nbhd U of $\mathbf{x}_0$ and a continuous map

$$G: U \to GL(J^{sr}(k))$$

such that, for every $\mathbf{x} \in U$:

i) $l(\mathbf{x})|J^r(k): J^r(k) \to E$ is an isomorphism.

ii) $G(\mathbf{x})|J^r(k)$ is an isomorphism onto $J^r(k)$.

iii) $G(\mathbf{x})|Ker(\pi_{sr.r})$ is an isomorphism onto $Ker(l(\mathbf{x}))$.

By shrinking, if necessary, we can assume that

$$\max_{x \in U} \{\| G(\mathbf{x})\|, \| G^{-1}(\mathbf{x})\|\} \leq L < +\infty.$$

Next, let

$$J^r(k) \xleftarrow{p_1} J^{sr}(k) \xrightarrow{p_2} Ker(\pi_{sr.r})$$

where $p_1 = \pi_{sr.r}$, and

$$J^r(k) \xleftarrow{q_1(\mathbf{x})} J^{sr}(k) \xrightarrow{q_2(\mathbf{x})} Ker(l(f(\mathbf{x}))), \ \mathbf{x} \in U$$

are the canonical projections. Clearly, q_1 and q_2 vary continously with their argument.

We note that, if $h: U \to J^{sr}(k)$ is any map, then

$$\begin{aligned} \pi_{sr.r}(h(\mathbf{x})) &= G^{-1}(\mathbf{x})(q_1(\mathbf{x}).h(\mathbf{x})) \\ p_2(h(x)) &= G^{-1}(x(q_2(x.h(x)))) \end{aligned} \qquad (5.1)$$

Now, we are ready to finish the proof of claim (5.3). Since $g_n \circ f \to 0$ in $C^{sr}(\bar{U})$, $j^{sr}(g_n \circ f) \to 0$ in $C^0(\bar{U},J^{sr}(k))$ and also

$$pr(j^{sr}(g_n \circ f)) = pr((f^*_{\mathbf{x}})_{sr}(j^{sr}g_n(f(\mathbf{x}))) \longrightarrow 0$$

C^0-uniformly over $\overline{U}$. This fact necessarily implies that

$$q_1(x).j^{sr}g_n(f(x)) \longrightarrow 0$$

C^0-uniformly over $\overline{U}$. This last convergence is clearly the same as: $j^r g_n(y) \to 0$ C^0-uniformly over $f(\overline{U})$, which proves the claim.

We turn now to the last tool necessary to prove theorem 2.1.

Claim 5.4. $Im((\tau_{k.d})^*)$ is dense in $C^\infty(\mathbf{C}^k \times \mathbf{R}^d)^{S_k}$.

Since the set of S_k-equivariant polynomials $P(\mathbf{C}^k \times \mathbf{R}^d)^{S_k}$ is dense in $C^\infty(\mathbf{C}^k \times \mathbf{R}^d)^{S_k}$, it suffices to show that this set is included in $Im((\tau_{k.d})^*)$.

An S_k-equivariant polynomial in $\mathbf{C}^k \times \mathbf{R}^d$ has the form:

$$\sum_{a=0}^{p} g_a(\mathbf{z},\overline{\mathbf{z}}).\mathbf{x}^a \quad , \quad \mathbf{z} \in \mathbf{C}^k \quad \text{and} \quad \mathbf{x} \in \mathbf{R}^d$$

where each g_a is an S_k-equivariant polynomial over $\mathbf{C}^k$. Therefore, without loss of generality, we can suppose d=0, and we shall now admit complex-valued polynomial mappings in variables $\mathbf{z},\overline{\mathbf{z}}$. We denote the ring of such maps by $P_{\mathbf{R}}(\mathbf{C}^k,\mathbf{C})$ and by $P(\mathbf{C}^k)$ the ring of holomorphic polynomials over $\mathbf{C}^k$.

It immediately becomes apparent that

$$\tau_k^* : P(\mathbf{C}^k) \longrightarrow P(\mathbf{C}^k)^{S_k}$$

is a ring isomorphism. To see that τ^* is surjective, pick any $p \in P(\mathbf{C}^k)^{S_k}$. The necessarily continuous function $\overline{p}$ which satisfies $p = \overline{p} \circ \pi_k$ is holomorphic in a thin set (the critical values of π_k), and then, p is holomorphic all over $\mathbf{C}^k$. Also, it must be a polynomial, as one can easily see.

Noting that

$$P_{\mathbf{R}}(\mathbf{C}^k,\mathbf{C}) = P(\mathbf{C}^k) \otimes_{\mathbf{C}} \overline{P(\mathbf{C}^k)}$$

and

$$P_{\mathbf{R}}(\mathbf{C}^k,\mathbf{C})^{S_k} = P(\mathbf{C}^k)^{S_k} \otimes_{\mathbf{C}} \overline{P(\mathbf{C}^k)}^{S_k}$$

where the overbar is conjugation, the proof of claim 5.4 and of theorem 2.1 can be easily finished.

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Recibido en febrero de 1989.