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TWO-DIMENSIONAL REAL DIVISION ALGEBRAS WITH EXACTLY ONE IDEMPOTENT

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ABSTRACT. We classify completely non-associative real division algebras of dimension two with exactly one idempotent. Each algebra of this type is isomorphic to precisely one member of ten infinite families.

In this paper we take "algebra" to mean a non-associative algebra over the field of real numbers **R**, that is, a real vector space A, with a product which is distributive vis-a-vis addition, and satisfies $(\alpha x)y = x(\alpha y) = \alpha(xy)$ for all $\alpha \in \mathbf{R}$, $x,y \in A$. An algebra is called a division algebra if the equation in A $ax = b$ (resp., $xa = b$) has unique solution whenever $a \neq 0$.

The fundamental work of Milnor and Bott [5], as well as that due to Kervaire [4], showed that all such finite-dimensional division algebras have dimension 1, 2, 4 or 8. Here we classify completely those of the dimension two which have exactly one idempotent. The algebra of the complex numbers, naturally, appears in this classification. That is the only algebra with unit element (see [3] for algebras without unit element of dimension 4 and 8).

An algebra A is called flexible if $(xy)x = x(yx)$ for all $x,y \in A$. Finite-dimensional flexible division algebras are classified in [1]. Our classification includes non-associative flexible algebras.

Let $[x_1,x_2]$ be a basis of a two dimensional algebra A. The product in A is determined by the multiplication table

(1)

	x_1	x_2
x_1	$\varphi x_1+\chi x_2$	$\psi x_1+\omega x_2$
x_2	$\alpha x_1+\beta x_2$	$\gamma x_1+\delta x_2$

The following result appears in [4]. (Theorem 1)

THEOREM. *The algebra* A *determined by* (1) *is a division algebra if and only if*

$$(2)\quad 4(\varphi\omega-\chi\psi)(\alpha\delta-\beta\gamma) > (\alpha\omega+\varphi\delta-\beta\psi-\chi\gamma)^2$$

$$(3)\quad 4(\varphi\beta-\alpha\chi)(\psi\delta-\gamma\omega) > (\alpha\omega-\varphi\delta-\beta\psi+\chi\gamma)^2$$

A non zero element x of an algebra A, is idempotent if $x^2 = x$. If A is a division algebra, A has one idempotent if and only if there exists $x \in A$, $x \neq 0$ and $c \in \mathbf{R}$: $x^2 = cx$. If the division algebra A has exactly one idempotent x_1, the algebra A' has one idempotent y_1, and $T: A \to A'$ is an isomorphism then $T(x_1) = y_1$.

We begin our study with the following lemma.

LEMMA 1. *Every two dimensional real division algebra* A *contains an idempotent.*

Proof. Let A be the two dimensional real division algebra determined by (1). If $\gamma = 0$, then, because of (2), $\delta \neq 0$ and $x = \frac{1}{\delta}x_2$ is an idempotent. If $\gamma \neq 0$, let $x = x_1+\xi x_2$.

$x^2 = cx$ for some $c \in \mathbf{R}$ if and only if

$$\gamma\xi^2+(\psi+\alpha)\xi+\varphi = c \text{ and } \chi+(\omega+\beta)\xi+\delta\xi^2-c\xi = 0$$

This system has a solution if and only if

$$\gamma\xi^3+(\alpha+\psi-\delta)\xi^2+(\varphi-\omega-\beta)\xi-\chi = 0$$

has a solution. Since every cubic polynomial possesses a root in $\mathbf{R}$, there exists some $\xi \in \mathbf{R}$ which satisfies this equation.

For an algebra with idempotent x_1, let $[x_1,x_2]$ be a basis. Hence the multiplication table has the form:

(4)

	x_1	x_2
x_1	x_1	$\psi x_1+\omega x_2$
x_2	$\alpha x_1+\beta x_2$	$\gamma x_1+\delta x_2$

Let $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ be the algebra with this multiplication table. By (2) and (3) it is a division algebra if and only if

$$4\omega(\alpha\delta-\beta\gamma) > (\alpha\omega+\delta-\beta\psi)^2 \tag{5}$$

All the idempotents of the division algebra $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ can be obtained provided we find ξ satisfying $(x_1+\xi x_2)^2 = c(x_1+\xi x_2)$ for some $c \in \mathbf{R}$, or by finding η such that $(\eta x_1+x_2)^2 = c'(\eta x_1+x_2)$ for some $c' \in \mathbf{R}$.

Now $(x_1+\xi x_2)^2 = c(x_1+\xi x_2)$ for some $c \in \mathbf{R}$, if and only if

(6) $\quad \gamma\xi^3+(\alpha+\psi-\delta)\xi^2+(1-\omega-\beta)\xi = 0$ has a solution.

$(\eta x_1+x_2)^2 = c'(\eta x_1+x_2)$ for some $c' \in \mathbf{R}$, if and only if

(7) $\quad (1-\omega-\beta)\eta^2+(\alpha+\psi-\delta)\eta+\gamma = 0$ has a solution.

Next we prove a necessary and sufficient condition in order that a division algebra have exactly one idempotent.

LEMMA 2. *Let* $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ *be a real division algebra with* $\omega+\beta = 1$ *and* $\psi+\alpha-\delta = 0$. *Then* $\gamma \neq 0$.

Proof. If $\delta = 0$, then $\gamma \neq 0$, for otherwise contradicting (5). Let $\delta \neq 0$ and suppose that $\gamma = 0$. The change of basis $y_1 = x_1$, $y_2 = x_1 - \frac{1}{\delta}x_2$ yields $y_2^2 = 0$, a contradiction.

THEOREM 3. *Suppose that the algebra* A *determined by* (4) *is a division algebra. Then it has exactly one idempotent if and only if* $(\psi+\alpha-\delta)^2-4\gamma(1-\omega-\beta) < 0$ *or* $\omega+\beta = 1$ *and* $\psi+\alpha-\delta = 0$.

Proof. If the algebra A has exactly one idempotent, then equation (6) has a unique solution $\xi = 0$.
But, (6) has a unique solution $\xi = 0$ if and only if $(\psi+\alpha-\delta)^2-4\gamma(1-\omega-\beta) < 0$, or $\omega+\beta = 1$ and $\psi+\alpha-\delta = 0$.

If $(\psi+\alpha-\delta)^2-4\gamma(1-\omega-\beta) < 0$, then it does not exist η fulfilling equation (7). If $\omega+\beta = 1$ and $\psi+\alpha-\delta = 0$ it does not exist η fulfilling equation (7) because $\gamma \neq 0$ (Lemma 2).

The following results are useful for our discussion.

LEMMA 4. *ω and β are invariants among division algebras with exactly one idempotent, that is, if $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ and $A(\psi',\omega',\alpha',\beta',\gamma',\delta')$ are isomorphic division algebras with exactly one idempotent, then $\omega = \omega'$ and $\beta = \beta'$.*

Proof. Suppose $T:A \to A'$ is an isomorphism. Then the image of the idempotent, x_1, of A is the idempotent y_1 of A'. Let $T(x_2) = \pi y_1+\sigma y_2$ $(\sigma \neq 0)$. Then

$$(8) \qquad (\psi+\omega\pi)y_1+\omega\sigma y_2 = T(x_1x_2) = T(x_1)T(x_2) = (\pi+\sigma\psi')y_1+\omega'\sigma y_2$$

$$(9) \qquad (\alpha+\beta\pi)y_1+\beta\sigma y_2 = T(x_2x_1) = T(x_2)T(x_1) = (\pi+\sigma\alpha')y_1+\beta'\sigma y_2$$

Whence $\omega\sigma = \omega'\sigma$, $\beta\sigma = \beta'\sigma$, and $\omega = \omega'$ and $\beta = \beta'$.

LEMMA 5. *Let $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ be a real division algebra with exactly one idempotent, with $\omega+\beta = 1$ and $\psi+\alpha-\delta = 0$. If $A(\psi',\omega',\alpha',\beta',\gamma',\delta')$ is an algebra isomorphic to $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$, then $\omega'+\beta' = 1$ and $\psi'+\alpha'-\delta' = 0$.*

Proof. Suppose we have an isomorphism
$T:A(\psi,\omega,\alpha,\beta,\gamma,\delta) \longrightarrow A(\psi',\omega',\alpha',\beta',\gamma',\delta')$ with $T(x_1) = y_1$, $T(x_2) = \pi y_1+\sigma y_2$. Then because of Lemma 4, $\omega'+\beta' = \omega+\beta = 1$.
From (8), (9) and

$$(\gamma+\delta\pi)y_1+\delta\sigma y_2 = T(x_2^2) = (T(x_2))^2 = (\pi^2+(\psi'+\alpha')\pi+\sigma\gamma')y_1+(\pi\sigma+\sigma^2\delta')y_2$$

we obtain $\alpha' = \dfrac{\alpha+\beta\pi-\pi}{\sigma}$; $\psi' = \dfrac{\psi+\omega\pi-\pi}{\sigma}$; $\delta' = \dfrac{\delta-\pi}{\sigma}$ and $\psi'+\alpha'-\delta' = 0$.

Our principal theorem follows.

THEOREM 6. *Let* A *be a real division algebra of dimension two with exactly one idempotent. Then* A *is isomorphic to precisely one of the following algebras:*

(class I) $A(\psi,\omega,1,\beta,\gamma,0)$ *for some* ψ,ω,β *and* γ *with* $\omega+\beta \neq 0$, $-4\omega\beta\gamma > (\omega-\beta\psi)^2$ *and* $(\psi+1)^2-4\gamma(1-\omega-\beta) < 0$,

or (class II) $A(|\psi|,\omega,0,\beta,1,0)$ *for some* ψ,ω *and* β *with* $\omega+\beta \neq 0$, $-4\omega\beta > \beta^2\psi^2$ *and* $\psi^2-4(1-\omega-\beta) < 0$,

or (class III) $A(\psi,\omega,0,\beta,-1,0)$ *for some* ψ,ω *and* β *with* $\omega+\beta \neq 0$, $4\omega\beta > \beta^2\psi^2$ *and* $\psi^2+4(1-\omega-\beta) < 0$,

or (class IV) $A(\psi,1,1,-1,\gamma,1)$ *for some* ψ *and* γ *with* $4\gamma > 4\psi+\psi^2$ *and* $\psi^2-4\gamma < 0$,

or (class V) $A(|\psi|,1,0,-1,1,0)$ *for some* ψ *with* $\psi^2 < 4$,

or (class VI) $A(0,-\beta,\alpha,\beta,\gamma,1)$ *for some* α,β *and* γ *with* $\beta \neq -1$, $-2\alpha\beta+4\beta^2\gamma > 1+\alpha^2\beta^2$ *and* $(\alpha-1)^2-4\gamma < 0$,

or (class VII) $A(0,-\beta,|\alpha|,\beta,1,0)$ *for some* α *and* β *with* $\beta \neq -1$ *and* $\alpha^2 < 4$,

or (class VIII) $A(-1,1-\beta,1,\beta,\gamma,0)$ *for some* β *and* γ *with* $4(1-\beta)\beta\gamma+1 < 0$,

or (class IX) $A(0,1-\beta,0,\beta,1,0)$ *for some* β *with* $\beta(1-\beta) < 0$,

or (class X) $A(0,1-\beta,0,\beta,-1,0)$ *for some* β *with* $\beta(1-\beta) > 0$.

Note that **C** the algebra of the complex numbers, appears in class III whenever $\psi = 0$, $\omega = 1$ and $\beta = 1$. That is the only algebra in our list with a unit element, as well as the only associative one. The flexible division algebras which coincide with the commutative algebras ([1]-Theorem 1-4) appear in class I for $\psi = 1$ and $\omega = \beta$, in class III for $\psi = 0$ and $\omega = \beta$ and class X for $\beta = \frac{1}{2}$.

We prove Theorem 6 in three steps. We show, in Lemmas 7,10 and 13, that any algebra $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ is isomorphic to an algebra in our list. Then we prove that no two algebras in the same class are (non-trivially) isomorphic (Lemmas 8,11 and 14). Finally we state that an algebra of a class cannot be isomorphic to an algebra of another class (Lemmas 4,5,9,12,15 and 16).

LEMMA 7. *If* $\omega+\beta \neq 0$, *the division algebra* $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ *is isomorphic to either*

$A(\psi',\omega,1,\beta,\gamma',0)$ *for some* ψ' *and* γ' *with* $-4\omega\beta\gamma' > (\omega-\beta\psi')^2$

or $A(\psi',\omega,0,\beta,1,0)$ *for some* ψ' *with* $-4\beta\omega > \beta^2\psi'^2$

or $A(\psi',\omega,0,\beta,-1,0)$ *for some* ψ' *with* $4\beta\omega > \beta^2\psi'^2$.

Proof. Without loss of generality we can take $\delta = 0$. For if $\delta \neq 0$ the change of basis to $y_1 = x_1$, $y_2 = x_1+\varepsilon x_2$ $(\varepsilon = -\frac{(\omega+\beta)}{\delta})$ gives a new multiplication with $\delta = 0$. This works because $y_2^2 = (1+\psi\varepsilon+\alpha\varepsilon+\gamma\varepsilon^2)x_1+\varepsilon(\omega+\beta+\varepsilon\delta)x_2$ and $\omega+\beta+\varepsilon\delta = 0$.

Now, if $\alpha \neq 0$, then the transformation T defined by $T(x_1) = y_1$, $T(x_2) = \alpha y_2$, yields an isomorphism $A(\psi,\omega,\alpha,\beta,\gamma,0) \cong$ $\cong A(\psi',\omega,1,\beta,\gamma',0)$ (with $\psi' = \frac{\psi}{\alpha}$ and $\gamma' = \frac{\gamma}{\alpha^2}$). This transformation T preserves products because

$$T(x_1x_2) = \psi y_1+\omega\alpha y_2 = \alpha\psi' y_1+\omega\alpha y_2 = T(x_1)T(x_2)$$
$$T(x_2x_1) = \alpha y_1+\alpha\beta y_2 = T(x_2)T(x_1) \quad \text{and}$$
$$T(x_2^2) = \gamma y_1 = \alpha^2\gamma' y_1 = (T(x_2))^2$$

On the other hand, if $\alpha = 0$, the transformation T given by $T(x_1) = y_1$, $T(x_2) = \sqrt{|\gamma|}\, y_2$ produces an isomorphism $A(\psi,\omega,0,\beta,\gamma,0) \cong A(\psi',\omega,0,\pm 1,0)$ with $\psi' = \frac{\psi}{\sqrt{|\gamma|}}$.

LEMMA 8. a) *The division algebra* $A(\psi,\omega,1,\beta,\gamma,0)$ *with* $\omega+\beta \neq 0$ *and with exactly one idempotent is isomorphic to the division algebra* $A(\psi',\omega,1,\beta,\gamma',0)$ *if and only if* $\psi = \psi'$ *and* $\gamma = \gamma'$.

b) *The division algebra* $A(\psi,\omega,0,\beta,1,0)$ *with* $\omega+\beta \neq 0$ *and with exactly one idempotent is isomorphic to the division algebra* $A(\psi',\omega,0,\beta,1,0)$ *if and only if* $\psi' = \pm\psi$.

c) *The division algebra* $A(\psi,\omega,0,\beta,-1,0)$ *with* $\omega+\beta \neq 0$ *and with* *exactly one idempotent is isomorphic to the division algebra* $A(\psi',\omega,0,\beta,-1,0)$ *if and only if* $\psi' = \psi$.

Proof. b) Suppose we have an isomorphism

$$T: A(\psi,\omega,0,\beta,1,0) \longrightarrow A(\psi',\omega,0,\beta,1,0)$$

with $T(x_1) = y_1$, $T(x_2) = \pi x_1+\sigma x_2$ $(\sigma \neq 0)$. Then

$$(\psi+\omega\pi)y_1+\omega\sigma y_2 = T(x_1x_2) = T(x_1)T(x_2) = (\pi+\sigma\psi')y_1+\omega\sigma y_2$$

$$y_1 = T(x_2^2) = (T(x_2))^2 = (\pi^2+\pi\sigma+\sigma^2)y_1+(\pi\sigma\omega+\pi\sigma\beta)y_2$$

Hence $\psi+\omega\pi = \pi+\sigma\psi'$, $\pi^2+\pi\sigma+\sigma^2 = 1$, $\pi\sigma(\omega+\beta) = 0$.
Since $\sigma \neq 0$ and $\omega+\beta \neq 0$, $\pi = 0$. Then $\sigma^2 = 1$ and $\psi' = \pm\psi$.

Conversely, the transformation T given by $T(x_1) = y_1$, $T(x_2) = -y_2$ produces an isomorphism $A(\psi,\omega,0,\beta,1,0) \cong A(-\psi,\omega,0,\beta,1,0)$.
In a similar way we can prove a) and c).

LEMMA 9. *An algebra* A *in the class* J *cannot be isomorphic to any algebra* A' *in the class* K ($J \neq K$, $J,K = I,II,III$).

Proof. Let $(J,K) = (I,II)$. If $T: A(\psi,\omega,1,\beta,\gamma,0) \longrightarrow A(\psi',\omega,0,\beta,1,0)$ is an isomorphism with $T(x_1) = y_1$, $T(x_2) = \pi y_1+\sigma y_2$, then

$$(1+\beta\pi)y_1+\beta\sigma y_2 = T(x_2x_1) = T(x_2)T(x_1) = \pi y_1+\sigma\beta y_2, \quad \text{and}$$

$$\gamma y_1 = T(x_2^2) = (T(x_2))^2 = (\pi^2+\pi\sigma\psi+\sigma^2)y_1+(\pi\sigma\omega+\pi\sigma\beta)y_2$$

Hence $1+\beta\pi = \pi$ and $\pi\sigma(\omega+\beta) = 0$. Since $\sigma \neq 0$ and $\omega+\beta \neq 0$, $\pi = 0$. Thus $1 = 0$, a contradiction.

The cases $(J,K) = (I,III)$ and $(J,K) = (II,III)$ are proved similarly.

LEMMA 10. *The division algebra* $A(\psi,1,\alpha,-1,\gamma,\delta)$ *is isomorphic to either* $A(\psi',1,1,-1,\gamma',1)$ *for some* ψ' *and* γ' *with* $4\gamma' > 4\psi'+\psi'^2$ *or* $A(\psi',1,0,-1,1,0)$ *for some* ψ' *with* $4 > \psi'^2$.

Proof. If $\delta \neq 0$, the transformation T given by $T(x_1) = y_1$, $T(x_2) = \delta y_2$ produces an isomorphism

$A(\psi,1,\alpha,-1,\gamma,\delta) \cong A(\psi',1,\alpha',-1,\gamma',1)$ (with appropriate ψ', α' and γ'). Furthermore, if $\alpha' \neq 1$, $A(\psi',1,\alpha',-1,\gamma',1) \cong$ $\cong A(\psi'',1,1,-1,\gamma'',1)$ using the transformation defined by $T(x_1) =$ $= y_1$, $T(x_2) = \frac{1}{2}(\alpha'-1)y_1+y_2$.

Finally, if $\delta = 0$, the transformation defined by $T(x_1) = y_1$, $T(x_2) = \frac{1}{2}\alpha y_1 + \frac{1}{2}\sqrt{4\gamma-\alpha^2-2\alpha\psi}$ produces an isomorphism $A(\psi,1,\alpha,-1,\gamma,0) \cong A(\psi',1,0,-1,1,0)$ with $\psi' = \dfrac{2\psi}{\sqrt{4\gamma-\alpha^2-2\alpha\psi}}$

LEMMA 11. a) *The division algebra* $A(\psi,1,1,-1,\gamma,1)$ *with exactly one idempotent is isomorphic to the division algebra* $A(\psi',1,1,-1,\gamma',1)$ *if and only if* $\psi = \psi'$ *and* $\gamma = \gamma'$.

b) *The division algebra* $A(\psi,1,0,-1,1,0)$ *with exactly one idempotent is isomorphic to the division algebra* $A(\psi',1,0,-1,1,0)$ *if and only if* $\psi' = \pm\psi$.

Proof. Similar to the proof of Lemma 8.

LEMMA 12. *An algebra* A *in the class* IV *cannot be isomorphic to any algebra* A' *in the class* V.

Proof. If $T: A(\psi,1,1,-1,\gamma,1) \longrightarrow A(\psi',1,0,-1,1,0)$ is an isomorphism with $T(x_1) = y_1$, $T(x_2) = \pi y_1+\sigma y_2$, then

$$(\gamma+\pi)y_1+\sigma y_2 = T(x_2^2) = (T(x_2))^2 = (\pi^2+\pi\sigma\psi'+\sigma^2)y_1$$

Hence $\sigma = 0$, a contradiction.

LEMMA 13. *The division algebra* $A(\psi,\omega,\alpha,\beta,\gamma,\delta)$ *with* $\omega+\beta = 0$ *and* $\omega \neq 1$ *is isomorphic to either* $A(0,-\beta,\alpha',\beta,\gamma',1)$ *for some* α' *and* γ' *with* $-2\beta\alpha'+4\beta^2\gamma' > \alpha'^2\beta^2+1$, *or* $A(0,-\beta,\alpha',\beta,1,0)$ *for some* α' *with* $4 > \alpha'^2$.

Proof. The change of basis $y_1 = x_1$, $y_2 = \frac{-\psi}{1+\beta}x_1+x_2$ gives a new multiplication table with $\psi = 0$.

Now, if $\delta \neq 0$, the transformation T defined by $T(x_1) = y_1$, $T(x_2) = \delta y_2$ yields an isomorphism $A(0,-\beta,\alpha,\beta,\gamma,\delta) \cong A(0,-\beta,\alpha',\beta,\gamma',1)$

(with $\alpha' = \frac{\alpha}{\delta}$ and $\gamma' = \frac{\gamma}{\delta^2}$). If $\delta = 0$, the transformation T given by $T(x_1) = y_1$, $T(x_2) = \sqrt{|\gamma|}\, y_2$ produces an isomorphism $A(0,-\beta,\alpha,\beta,\gamma,0) \cong A(0,-\beta,\alpha',\beta,1,0)$ with $\alpha' = \frac{\alpha}{\sqrt{|\gamma|}}$

LEMMA 14. a) *The division algebra* $A(0,-\beta,\alpha,\beta,\gamma,1)$ *with* $\beta \neq -1$ *and with exactly one idempotent is isomorphic to the division algebra* $A(0,-\beta,\alpha',\beta,\gamma',1)$ *if and only if* $\alpha = \alpha'$ *and* $\gamma = \gamma'$.

b) *The division algebra* $A(0,-\beta,\alpha,\beta,1,0)$ *with* $\beta \neq -1$ *and with exactly one idempotent is isomorphic to the division algebra* $A(0,-\beta,\alpha',\beta,1,0)$ *if and only if* $\alpha' = \pm\alpha$.

Proof. Similar to the proof of Lemma 8.

LEMMA 15. *An algebra* A *in the class* VI *cannot be isomorphic to any algebra* A' *in the class* VII.

Proof. Similar to the proof of Lemma 9.

Finally

LEMMA 16. *An algebra* A *in the class* J *cannot be isomorphic to any algebra* A' *in the class* K ($J \neq K$ J,K = VIII,IX,X).

Proof. Similar to the proof of Lemma 9.

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