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ON THE EXISTENCE OF LOCALLY HEAVY ARCS

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INTRODUCTION

Everyone is familiar with space's filling curves 'a la' Peano. Those curves are continuous functions of [0,1] but, by the Invariance of domain theorem, they are barred from being one-to-one. On the other hand, an arc is a homeomorphism of [0,1] into a metric space. Thus, it is reasonable to expect that arcs are unidimensional objects. This is indeed the case if dimension is understood in the topological sense as in Hurewicz and Wallman [5, p. 24]. However, if ν is a square's filling curve, the arc $\tau(t) = (t, \nu(t))$ has a 2-dimensional projection; i.e., it has a heavy shadow. In 1903 W.F. Osgood [7] showed that there exists a closed Jordan curve with positive exterior area (actually his curve has positive plane measure [2, p. 138]. Osgood's paper, which was motivated by a question of Jordan, makes for enjoyable reading; as an additional bonus, it has two beautiful color pictures (at least in the 1964 reprint). Another fascinating curve was discovered by Koch in 1904; for pictures of it, see [6, pp. 42-43]. Mandelbrot, in discussing the Koch-type curves, mentioned that they can be constructed with any possible Hausdorff dimension between 1 and 2, [6, p. 39]. The regularity used in the construction would mean that any portion of a Koch-type curve has the same Hausdorff dimension.

Recently, T. Lance and E. Thomas [3] have constructed a one parameter family of arcs in R^2 with positive plane Lebesgue measure which converge to a space filling curve. In section 2, we show that the Lance and Thomas's construction, after a suitable modification, still yields the fact that there exist arcs which have positive plane Lebesgue measure wherever you look at them. Actually, for q such that $1 < q \leq 2$, the same construction also yields the above mentioned fact that there exists an arc in R^2 such that any subarc has Hausdorff dimension q.

In section 3, we exhibit an arc which is "infinitely heavy" in the sense that its Hausdorff dimension is greater than any prescribed number. Such an arc cannot, of course, live in a finite dimensional space. Ours lives in an infinite-dimensional separable Hilbert space, and while in it we'll see that "the crinkled arc" $f : [0,1] \longrightarrow L^2([0,1], dx)$, defined by $f(t) = \chi_{[0,t]}$, has Hausdorff dimension 2. As mentioned in Halmos [3, problem 5], every pair of disjoint subarcs of the crinkled arc are perpendicular; hence its name.

LOCALLY HEAVY ARCS

Let E be a metric space. We recall how its Hausdorff dimension is defined [5. p. 105].

Let p be a real number and $c > 0$,

$$h_p^c(E) = inf\{\sum_{i=1}[d(E_i)]^p\}$$

where $E = \cup E_i$ is any partition of E in a countable number of subsets of diameter$(E) = d(E) < c$. Let

$$h_p(E) = sup\{h_p^c(E) : c > 0\}$$

$h_p(E)$ is called the Hausdorff p-measure of E. The Hausdorff dimension of E is

$$h(E) = sup\{p : h_p(E) > 0\}.$$

When E is compact, it suffices to consider finite partitions. The topological dimension of E satisfies:

$$dim(E) = inf\{h(E') : E' \textit{ is homeomorphic to } E\}$$

Remark 0. B. Mandelbrot [6, p. 15] suggests the term Hausdorff-Besicovitch dimension as being more historically accurate.

Remark 1. $h_q(E)$ finite implies that $h_p(E) = 0$ if $p > q$.

Remark 2. Let $f : E \longrightarrow E'$ be bounded; i.e., there exists $C > 0$ so that

$$dist(f(x), f(y)) \leq C dist(x, y).$$

Also, let f be onto. Then $h(E') \leq h(E)$.

Thus, the arc given in the introduction has $h(\tau[I]) \geq 2$, where I stands for $[0, 1]$.

The next known lemma will be used in the proof of the theorem.

Lemma 3. Let $1 < q < 2$ and let r be such that $4r^q = 1$. Let

$$A = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{4^k} A_k(i)$$

be a Cantor subset of the unit square I^2 such that,

1) $A_{k+1}(4(i-1)+v) \subset A_k(i)$, $v = 1, 2, 3, 4$; where each of these sets is a square with sides parallel to the x and y axis and the smaller squares contain the vertices of $A_k(i)$.
2) $A_k(i)$ and $A_k(j)$ are disjoint if $i \neq j$.
3) The size of $A_k(i)$ is r^k.

Then A has Hausdorff dimension q.

Remark 4. Let A be as in Lemma 3. If g is an adequate convex function that sends the unit square I^2 onto a diamond contained in I^2, then $h(g(A)) = q$. This follows from applying Remark 2 since g and g^{-1} are bounded.

Remark 5. Let $\lim_{k\to\infty} r_k = 1/2, r_k > 0$. If A is as in the lemma except that the side of the $A_i(v)'s$ are $\prod_{k=1}^{i} r_k$, then $h(A) = 2$.

In preparation for the theorem, we need the following sets: for each $n \in \mathbf{N}$, let

$$D_n = \{\mathbf{i} = (i_1, \ldots, i_n) : i_k \in \{0, 1, \ldots, 7\} \text{ } for \text{ } 1 \leq k \leq n\}$$

endowed with the lexicographic order. Let $(i_1, \ldots, i_n, j) \in D_{n+1}$ be denoted by $(\mathbf{i}, j)$, where $\mathbf{i} \in D_n$.

We recall that in each step of the Lance and Thomas's construction there are squares and segments. What we basically do is use diamonds instead of segments. When $1 < q \leq 2$, the diamonds are chosen in such a way that their *width* (length of their shorter diagonals) go to zero very rapidly, while the corresponding "squares" (in fact they are also diamonds) are chosen judiciously.

We emphasize again that, although the content of the next theorem is not new, its proof provides a unified and easy way of exhibiting arcs which locally have the same previously specified Hausdorff dimension.

Theorem 6. Let $1 < q \leq 2$. Then there exists an arc $\Gamma : [0, 1] \longrightarrow R^2$, such that for each $\Gamma(t)$ and each $\epsilon > 0$, $\{y \in R^2 : |\Gamma(t) - y| < \epsilon\} \bigcap \Gamma([0, 1])$ has Hausdorff dimension q. When $q = 2$, there exists an arc such that each subarc has positive two dimensional Lebesgue measure.

Proof: Let us consider first $q = 2$. We construct inductively nested sets B_k such that the arc satisfies

$$\Gamma[I] = \cap_{k=1}^{\infty} B_k$$

. Each B_k is the union of 8^k diamonds in which each pair of distinct diamonds intersects at most in one point. Every B_k is connected.

In the first step we construct $B_1 = \cup_{i=1}^{7} A(i)$ according to figure 1. B_1 is contained in I^2 and each of the squares $A(0), A(2), A(5), A(7)$ contains a vertex of I^2, and they are all translates of $A(0)$. Each square has side r_1.$A(1)$ is a translate of $A(6)$, while $A(3)$ is a translate of $A(4)$. (The 8 diamonds have positive two-dimensional Lebesgue measure.)

Let r_k be an increasing sequence of positive numbers with limit 1/2 such that

$$\lim_{k\to\infty} 2^k \prod_{j=1}^{k} r_j > 0. \tag{1}$$

Assume that we have already constructed $B_t, t = 1, \ldots, n$. Let

$$B_{n+1} = \bigcup_{\mathbf{i}\in\mathbf{D_{n+1}}} A(\mathbf{i})$$

where the A($\mathbf{i}$) satisfy:

1) If $\mathbf{j} \in D_n$, then $A(\mathbf{j}, k) \subset A(\mathbf{j})$,
2) Each $A(i_1, \ldots, i_{n+1})$, with $\{i_1, \ldots, i_{n+1}\} \subset \{0, 2, 5, 7\}$, is a square of side $\prod_{k=1}^{n+1} r_k$,
3) For each pair of distinct $\mathbf{i}, \mathbf{k} \in D_{n+1}$, $A(\mathbf{i})$ and $A(\mathbf{k})$ intersect at most in one point, and such a point is a common vertex.
4) If $\mathbf{j} \in D_n, A(\mathbf{j}, 1)$ is a translate of $A(\mathbf{j}, 6)$, while $A(\mathbf{j}, 3)$ is a translate of $A(\mathbf{j}, 4)$. The $A(\mathbf{j}, u)'s$ with $u = 0, 2, 5, 7,$ are all translate of each other. Also $d(A(\mathbf{j}, 7)) = r_{n+1} d(A(\mathbf{j}))$.
5) Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be three consecutive words of length n. Suppose that the diamonds $A(\mathbf{i}), A(\mathbf{j})$ and $A(\mathbf{k})$ are in the position shown in figure 2. Then the enumeration of $A(\mathbf{j}, u), u = 0, \ldots, 7$, is as shown in figure 3; i.e, the order is from $A(\mathbf{i})$ to $A(\mathbf{k})$. (Notice that the actual position of figure 2 may be rotated. When $\mathbf{j} = (0, \ldots, 0)$ we can have only $\mathbf{j} \leq \mathbf{k}$; while $\mathbf{j} = (7, \ldots, 7)$ admits only $\mathbf{i} \leq \mathbf{j}$.)

Let $x \in I$ be written in base 8, $x = \sum x_i / 8^i$. We define the arc $\Gamma : I \longrightarrow I^2$ by

$$\Gamma(x) \in \bigcap_{n=1}^{\infty} A(x_1, \ldots, x_n)$$

The set $\cap_{n=1}^{\infty} A(x_1, \ldots, x_n)$ is a singleton since the family $\{A(x_1, \ldots, x_n) : 1 \leq n\}$ is nested and $d(A(x_1, \ldots, x_n)) \leq 2^{-n/2}$. The only case when x can be written in two distinct forms is , for $1 \leq x_n$, if

$$x = \sum_{m=1}^{n} x_m / 8^m = \sum_{m=1}^{n-1} x_m / 8^m + (x_n - 1)/8^n + \sum_{m=n+1}^{\infty} 7/8^m.$$

But it is clear that $A(x_1, \ldots, x_{n-1}, x_n, 0, \ldots, 0)$ and $A(x_1, \ldots, x_{n-1}, x_n - 1, 7, \ldots, 7)$ have a common point. That Γ is one-to-one and continuous is clear.

Let $m_n(\ )$ denote n-dimensional measure. If $\mathbf{i} \in D_k$, then $m_2(A(\mathbf{i}, 0)) = r_{k+1}^2 m_2(A(\mathbf{i}))$. This and inequality (1) imply

$$m_2(A(\mathbf{i}) \cap \Gamma(I)) \geq \prod_{j=1}^{\infty} (2r_{n+j})^2 m_2(A(\mathbf{i})) > 0.$$

If we only ask that $\lim_{n \to \infty} r_n = 1/2$, then Lemma 3 and Remark 4 imply that $h(A(\mathbf{i}) \cap \Gamma(I) = 2$.

We now tackle the case $1 < q < 2$. Let r be such that $4r^q = 1$ and let us choose an increasing sequence of natural numbers n_k such that

$$\sum_{k=1}^{\infty} 8^k n_k^{1-q} < \infty, \tag{2}$$

and

$$r^k n_k > 1. \tag{3}$$

It will be convenient to denote $\{0, 2, 5, 7\}$ by S and $\{1, 3, 4, 6\}$ by T. The construction of the B_k is as before except that 2) is replaced by

2') $A(i_1,\ldots,i_k)$, with $\{i_1,\ldots,i_k\} \subset S$, is a square of side r_k,
and we also ask:

6) If $i_k \in T$, then the width of $A(i_1,\ldots,i_k)$ is at most $1/n_k$.

Conditions 2') and 4); Lemma 3, and Remark 4 imply that for all $\mathbf{i}'s$,

$$h(A(\mathbf{i}) \cap \Gamma(I)) \geq q.$$

Thus, it suffices to show, by Remark 1, that $h_q(\Gamma(I))$ is finite. In fact, it is at most $2(1+\sum_{k=1}^{\infty} 8^k n_k^{1-q})$.

Let $\epsilon > 0$ and j be so that $2^{1/2}r^j < \epsilon$. In what follows all the $\mathbf{i}$'s are in D_j.

If $\mathbf{i}$ is such that all of its components are in S; i,e,

$$\mathbf{i} \in \underbrace{S \times \ldots \times S}_{j \text{ times}},$$

then $d(A(\mathbf{i}) \leq 2^{1/2}r^j$. There are 4^j of these $\mathbf{i}'s$. If $\mathbf{i}$ is such that its g-component is in T, while the $g+1,\ldots,j$ components are in S, we partition

$$A(\mathbf{i}) = \bigcup_{u=1}^{n_g} X(\mathbf{i},u)$$

in such a way that the intersection of $X(\mathbf{i},u)$ with the long diagonal of $A(\mathbf{i})$ has length $1/n_g$ the length of such diagonal (see figure 4, assumig $n_g = 8$). Each X(**i**,u) has diameter at most $2^{1/2}(1/n_g)r^{j-g}$. Inequality (3) implies that $2^{1/2}(1/n_g)r^{j-g} < \epsilon$. (Observe that there are $2^{g-1}4^j$ of these $\mathbf{i}'s$.) The choice of r and of $\{n_k\}$ implies the following inequalities

$$\sum_{\mathbf{i} \in \underbrace{S \times \ldots \times S}_{j \text{ times}}} [d(A(\mathbf{i}))]^q \leq 2^{q/2}4^j r^{jq} = 2^{q/2}$$

and

$$\sum_{\mathbf{i} \notin \underbrace{S \times \ldots \times S}_{j \text{ times}}} \sum_{u=1}^{n_g} [d(X(\mathbf{i},u))]^q \leq 2^{q/2} \sum_{g=1}^{j} 2^{g-1}4^j n_g[(1/n_g)r^{j-g}]^q.$$

Thus the sum of the left-hand side in the two preceeding inequalities is at most

$$2^{q/2}[1+\sum_{g=1}^{\infty}(1/2)8^g n_g^{1-q}]$$

By using inequality (2), we see that this last quantity is finite. This shows that $h(\Gamma(I)) \leq q$, which is the only estimate needed to conclude the proof of the theorem.

Remark 7. This same construction works in higher dimensions; for each natural number $n > 1$, and $n-1 \leq q \leq n$, there exists an arc in I^n such that each subarc has Hausdorff dimension q.

INFINITELY HEAVY ARCS AND THE CRINKLED ARC

For each $n > 1$, let $\Phi_n : I \longrightarrow I^n$ be an n-cube's filling curve. Let H be the real Hilbert space $H = \oplus R^n$. The arc $\Phi : I \longrightarrow H$ defined by

$$\Phi(t) = (t, (1/2)\Phi_2(t), (1/3)\Phi_3(t), \ldots)$$

is infinitely heavy. This follows from Remark 2 since orthogonal projections are contractions. By using Remark 7, a little more can be said. Let Q be any denumerable dense subset of $[1, \infty)$. For each $q \in Q$, let $n_q = q$ if $q \in N$ and $n_q = [q] + 1$ otherwise ($[q] =$ integer part of q). There exists an arc Ω_q such that each subarc has Hausdorff dimension q. Let g be any injective mapping from N onto Q. For each j, let $H_j = R^{n_{g(j)}}$. Define the arc $\Omega : I \longrightarrow \oplus H_j$ by

$$\Omega(t) = (\Omega_{g(1)}(t), (1/2)\Omega_{g(2)}(t), (1/3)\Omega_{g(3)}(t), \ldots)$$

For each $j \in N$, let $P_{g(j)}$ be the orthogonal projection in H_j . Then for each interval $[a, b] \subset I$, $P_{g(j)}(\Omega[a, b])$ has Hausdorff dimension q. It is plausible that for each $x \in [1, \infty)$ and every $[a, b]$, there exists a finite dimensional projection P_x such that

$$h(P_x(\Omega([a, b]))) = x$$

In what follows we'll consider that $f : I \longrightarrow L^p(I, dx)$ is defined by $f(t) = \chi_{[0,t]}$, for $1 < p < \infty$. We'll still call it the crinkled arc.

Proposition 8. The crinkled arc has Hausdorff dimension p.

Proof. Since $d(f([a, b]) = (b-a)^{1/p}, \sum_{i=1}^n d(f([(i-1)/n, i/n])^p = 1$. Thus Remark 1 implies that $h(f(I)) \leq p$.

Let $1 < q < p$. We now show that $h_q(f(I)) = \infty$. Observe that $B \subset f(I)$ implies that $d(f^{-1}(B)) = d(B)^p$.

Let $f(I) = \cup_{i=1}^n B_i$ with $d(B_i) \leq \epsilon$. Since

$$1 = m_1(\cup_{i=1}^n f^{-1}(B_i)) \leq \sum_{i=1}^n m_1(f^{-1}(B_i)) \leq \sum_{i=1}^n d(f^{-1}(B_i)),$$

it follows that

$$\sum_{i=1}^n d(B_i)^q = \sum_{i=1}^n d(f^{-1}(B_i))^{q/p} \geq \epsilon^{q-p} \sum_{i=1}^n d(f^{-1}(B_i)) \geq \epsilon^{q-p}.$$

Thus $h_q(f(I)) = \infty$ and so $h(f(I)) = p$.

Remark 9. The crinkled arc looks exactly the same (except at the end points, of course) wherever you look at it.

Proposition 10. The projection of the crinkled arc in any (non-zero) finite dimensional subspace of $L^p(I, dx)$ has Hausdorff dimension 1.

Proof. Let $P : L^p(I, dx) \longrightarrow L^p(I, dx)$ be a finite dimensional projection. Then there exist linearly independent functions $g_1, \ldots, g_m$ in $L^p(I, dx)$ and linearly independent functions $k_1, \ldots, k_m$ in $L^q(I, dx)$, with $1/p + 1/q = 1$, so that for $r \in L^p(I, dx)$

$$P(r) = \sum_{j=1}^{m} \Big(\int_0^1 r(x) k_j(x) dx \Big) g_j.$$

For each $n \in N$ express $P(f(I))$ as $\cup_{i=1}^{n} P(f([(i-1)/n, i/n]))$. The definition of the the diameter of $P(f([(i-1)/n, i/n]))$ is

$$sup\{\|P(f(s)) - P(f(t))\|_p : (i-1)/n \leq s, t \leq i/n\}.$$

Since

$$\|P(f(s)) - P(f(t))\|_p \leq \sum_{j=1}^{m} \Big(\int_{i/n}^{(i-1)/n} |k_j(x)| dx \Big) \|g_j\|_p,$$

it follows that

$$d(P(f([(i-1)/n, i/n])) \leq \sum_{j=1}^{m} \Big(\int_{i/n}^{(i-1)/n} |k_j(x)| dx \Big) \|g_j\|_p.$$

Therefore

$$\sum_{i=1}^{n} d(P(f([(i-1)/n, i/n]))) \leq \sum_{j=1}^{m} \Big(\int_0^1 |k(x)| dx \Big) \|g_j\|_p$$

and, by Holder's inequality, the last quantity is at most

$$\sum_{j=1}^{m} \|k_j\|_q \|g_j\|_p < \infty.$$

Thus $h(P(f(I)) \leq 1$. To see the other inequality it suffices to consider, by Remark 2, the case when P is a one dimensional projection $P(r) = \big(\int_0^1 r(x) k(x) \big) g$.

We may assume that the real part of the function k is non-zero. Therefore the real part of $\{\int_0^s k(x) dx : s \in [0, 1]\}$ is a non-degenerate interval of R. This concludes the proof of the proposition.

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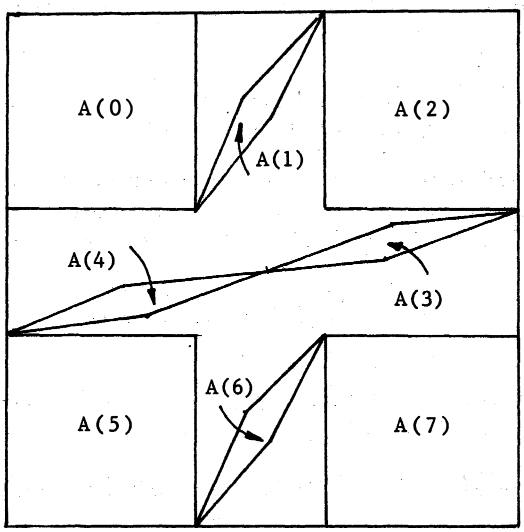

Figure 1

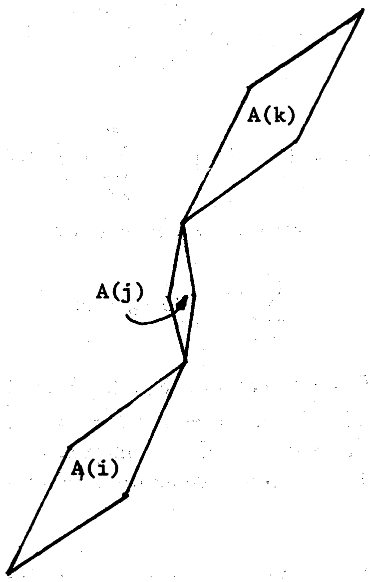

Figure 2

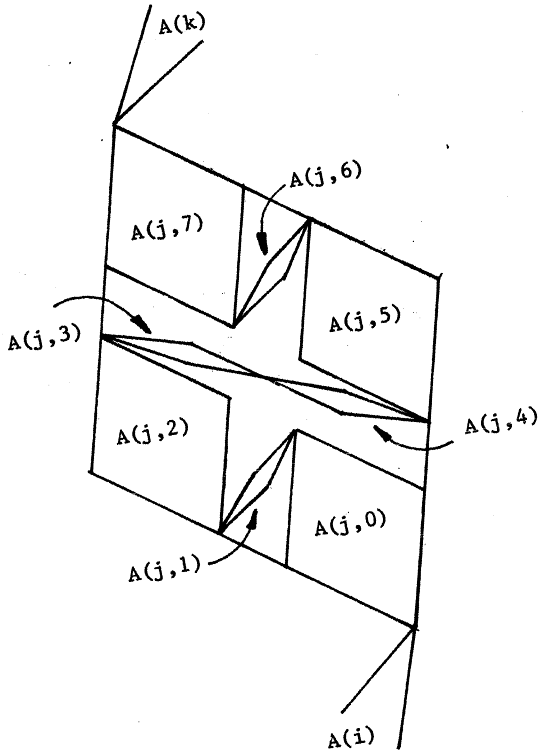

Figure 3

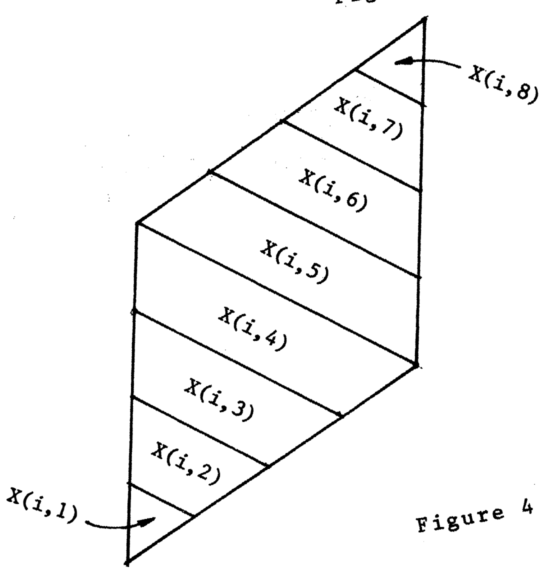

Figure 4

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