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A NOTE ON ITERATED SIMILARITIES OF OPERATORS

DOMINGO A. HERRERO (*)

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In [3], J.A.Deddens introduced the set

$$\mathcal{B}_A = \{X \in L(H): \sup_{n\geq 1} \|A^n X A^{-n}\| < \infty\}$$

associated with an invertible operator A. ($L(H)$ is the algebra of all bounded linear operators acting on the complex, separable, infinite dimensional Hilbert space H.) $\mathcal{B}_A$ is an algebra that includes the commutant $A'(A)$ of A.

The question (raised by Deddens in [3]) of whether $\mathcal{B}_A = L(H)$ implies that A is a non-zero multiple of a similarity of a unitary operator has been affirmatively answered, independently, by D.A.Herrero [5], J.G.Stampfli [9] and J.P.Williams [11]. Indeed, Stampfli and Williams merely assume that $\mathcal{B}_A \supset K(H)$, the ideal of all compact operators. The first result of this note says that the same conclusion is true if we just assume that $\mathcal{B}_A$ contains all the rank-one operators. The second result extends a theorem of J.P.Williams [11] by showing that $\mathcal{B}_A \cap \mathcal{B}_{A^{-1}} = A'(A)$ for a large family of operators.

The third and last result is an example showing that we can have

$$\sup_{n\geq 1} \|A^n X A^{-n}\| < \infty \quad \text{and} \quad \|A^n R A^{-n}\| \to 0 \quad (|n| \to \infty)$$

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for a suitably chosen non-zero nilpotent operator R and A invertible. This negatively answers a question of C.M.Pearcy (personal communication).

THEOREM 1. *The following are equivalent for* A *in* $L(H)$:

(i) $B_A = L(H)$.

(ii) $B_A \supset K(H)$.

(iii) B_A *contains all rank-one operators.*

(iv) $A = rWUW^{-1}$, *where* $r > 0$, W *is invertible and* U *is unitary.*

Proof. (iv) ⇒ (i) ⇒ (ii) ⇒ (iii) are trivial implications. Assume that B_A contains all operators of the form $f \otimes g^*$ ($f \otimes g^*(x) = \langle x,g\rangle f$, $x \in H$). Observe that $A^n(f \otimes g^*)A^{-n} = (A^n f) \otimes (A^{*-n}g)^*$. Fix g; since $f \otimes g^* \in B_A$, the function

$$f \to a_g(f) := \sup_{n\geq 0} \|A^n(f \otimes g^*)A^{-n}\| = \sup_{n\geq 0} \|A^n f\| . \|A^{*-n}g\|$$

is *finite* for all $f \in H$. It is completely apparent that a_g is a Borel function defined on H with values in $\mathbf{R}^+$, $a_g(f_1 - f_2) \leqslant a_g(f_1) + a_g(f_2)$ and $a_g(tf) = ta_g(f)$ for $t > 0$. Thus, by Helson's uniform boundedness principle [4,Theorem 1], there is a positive constant K(g) such that $a_g(f) \leqslant K(g)\|f\|$ for all $f \in H$.

Furthermore, K(g) can be chosen equal to

$$K(g) = \sup_{n\geq 0} \sup_{\|f\|=1} \|A^n f\| . \|A^{*-n}g\| = \sup_{n\geq 0} \|A^n\| . \|A^{*-n}g\| ,$$

and the function $g \to K(g)$ has exactly the same properties as a_g. Thus, a new application of Helson's uniform boundedness principle implies that $K(g) \leqslant C\|g\|$ for all $g \in H$ and some constant $C > 0$.

Let $r = |\lambda|$ for some λ in the spectrum, $\sigma(A)$, of A. We conclude from the above argument that

$$\|(r^{-1}A)^n\| . \|(r^{-1}A)^{-n}\| = \|A^n\| . \|A^{-n}\| = \sup_{\|f\|=\|g\|=1} \|A^n f\| . \|A^{*-n} g\| \leqslant C$$

for all $n \geqslant 0$. A fortiori, $1 \leqslant \|(r^{-1}A)^n\| \leqslant C$ for all *integers* n. By a well-known theorem of B.Sz.-Nagy [10], it follows that

$$A = rWUW^{-1}$$

for some invertible W and some unitary U.

The proof of Theorem 1 is now complete. ∎

REMARK. The conclusion "$\|(r^{-1}A)^n\| \leqslant C$ for all integers n" holds for every (real or complex) Banach space.

J.P.Williams has shown that if Q is quasinilpotent, then $\mathcal{B}_{1+Q} \cap \mathcal{B}_{(1+Q)^{-1}} = \mathcal{A}'(Q)$ [11]. His result admits the following mild extension.

THEOREM 2. *Assume that* $A \in \mathcal{L}(\mathcal{H})$ *is an invertible operator with totally disconnected spectrum*

$$\sigma(A) \subset \{r\ e^{i\phi(r)}: r > 0\}$$

for some real-valued function ϕ *defined on* $(0,\infty)$*; then*

$$\mathcal{B}_A \cap \mathcal{B}_{A^{-1}} = \mathcal{A}'(A).$$

Proof. Observe that $A = e^S$, where $S = \log A$ for some branch of log analytic on $\sigma(A)$; moreover, the weak closure of the polynomials in S coincides with the weak closure of the polynomials in A, and therefore $\mathcal{A}'(A) = \mathcal{A}'(S)$. Since $\sigma(A)$ and $\sigma(S)$ are totally disconnected, given $\varepsilon > 0$ we can write

$$A = W_\varepsilon(\textstyle\sum \oplus_{j=1}^m A_j)W_\varepsilon^{-1} \quad , \qquad S = W_\varepsilon(\textstyle\sum \oplus_{j=1}^m S_j)W_\varepsilon^{-1} \ ,$$

where $\{A_j\}_{j=1}^m$ and $\{S_j = \log A_j\}_{j=1}^m$ $(m = m(\varepsilon))$ are finite families of operators with pairwise disjoint spectra such that diameter $\sigma(A_j) < \varepsilon$ and diameter $\sigma(S_j) < \varepsilon$ for all $j = 1,2,\ldots,m$, and $sp(A_j)sp(A_{j+1}{}^{-1}) < 1$ for $j = 1,2,\ldots,m-1$. (Here sp(T) denotes the spectral radius of the operator T.)

Suppose $\|A^n X A^{-n}\| \leqslant C(X)$ for all $n \in \mathbf{Z}$; then

$$\sup_{\lambda \in \mathbf{R}} \|e^{\lambda S} X e^{-\lambda S}\| \leqslant (\max_{0 \le t \le 1} \|e^{tS}\|)(\max_{0 \le t \le 1} \|e^{-tS}\|) \sup_n \|e^{nS} X e^{-nS}\| =$$

$$= (\max_{0 \le t \le 1} \|e^{tS}\|)(\max_{0 \le t \le 1} \|e^{-tS}\|) \sup_n \|A^n X A^{-n}\| \leqslant C'(X).$$

Therefore, $f(\lambda) = e^{\lambda S} X e^{-\lambda S}$ defines an entire function of exponential type, with values in $L(H)$.

Observe that

$$A^n X A^{-n} = W_\varepsilon (\textstyle\sum \oplus_{j=1}^m A_j^{\,n})(W_\varepsilon^{-1} X W_\varepsilon)(\textstyle\sum \oplus_{j=1}^m A_j^{\,-n}) W_\varepsilon^{-1}.$$

Thus, if $W_\varepsilon^{-1} X W_\varepsilon = (X_{ij})_{i,j=1}^m$, then

$$\|W_\varepsilon^{-1}(A^n X A^{-n}) W_\varepsilon\| = \|(A_i^{\,n} X_{ij} A_j^{\,-n})_{i,j=1}^m\| \leqslant \|W_\varepsilon\| . \|W_\varepsilon^{-1}\| C(X)$$

for all n, whence we immediately see that $X_{ij} = 0$ for all $i \neq j$. Hence,

$$A^n X A^{-n} = W_\varepsilon (\textstyle\sum \oplus_{j=1}^m A_j^{\,n} X_{jj} A_j^{\,-n}) W_\varepsilon^{-1}.$$

Observe that $A = \exp S$, where $S = \log A = W_\varepsilon (\sum \oplus_{j=1}^m A_j) W_\varepsilon^{-1}$ for some branch of log analytic on $\sigma(A)$. It follows as in [11] that

$$\|e^{\lambda S} X e^{-\lambda S}\| = \|\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \delta_S^{\,n}(X)\| = \|W_\varepsilon (\textstyle\sum \oplus_{j=1}^m e^{\lambda S_j} X_{jj} e^{-\lambda S_j}) W_\varepsilon^{-1}\| \leqslant$$

$$\leqslant \|W_\varepsilon\| . \|W_\varepsilon^{-1}\| \max_j \|e^{\lambda S_j} X_{jj} e^{-\lambda S_j}\| =$$

$$= \|W_\varepsilon\| . \|W_\varepsilon^{-1}\| . \|\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \textstyle\sum \oplus_{j=1}^m \delta_{S_j}^{\,n}(X_{jj})\| ,$$

where $\delta_T(R) = TR - RT$ and $\delta_T^{\,n+1}(R) = \delta_T(\delta_T^{\,n}(R))$.

Therefore, for a sufficiently large N, we have

$$\|e^{\lambda S} X e^{-\lambda S}\| \leqslant \|W_\varepsilon\| . \|W_\varepsilon^{-1}\| \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} \|\textstyle\sum \oplus_{j=1}^m \delta_{S_j}^{\,n}(X_{jj})\| \leqslant$$

$$\leqslant \|W_\varepsilon\|^2 . \|W_\varepsilon^{-1}\|^2 . \|X\| \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} \left(\max_{1\le j\le m} \|\delta_{S_j}^{\ n}\| \right) \leqslant$$

$$\leqslant \|W_\varepsilon\|^2 . \|W_\varepsilon^{-1}\|^2 \|X\| \max_{1\le j\le m} \left\{ \sum_{n=0}^{N} \frac{|\lambda|^n}{n!} \|2S_j\|^n + \sum_{n=N+1}^{\infty} \frac{|2\varepsilon\lambda|^n}{n!} \right\} \leqslant$$

$$\leqslant C'_\varepsilon(X)\ e^{2\varepsilon|\lambda|} .$$

Hence, either the function f has order $\rho < 1$, or it has order $\rho = 1$ and type 0. In either case, since $\|f(\lambda)\|$ is bounded on the real axis, f is bounded everywhere, and therefore it is a constant function [1,10.2.1], [8,p.282].

This means, in particular, that

$$\delta_S(X) = SX - XS = 0 ;$$

that is, X commutes with S. A fortiori, $X \in A'(A)$.

PROBLEM 3. Suppose $A \in L(H)$ is an invertible operator such that

$$\sigma(A) \subset \Gamma(A) := \{r\ e^{i\phi(r)} : r > 0\}$$

for some real-valued function ϕ. (By using the compactness of $\sigma(A)$, we can easily see that ϕ can be chosen a continuous function.)

Does $B_A \cap B_{A^{-1}} = A'(A)$?.

Deddens [3] proved that this is the case if A is a positive hermitian operator. In fact, his results actually show that $B_A \cap B_{A^{-1}} = A'(A)$ for every *normal* operator A with $\sigma(A)$ included in an increasing arc $\Gamma(A)$ of the above described type. Theorem 1 shows that, in general, $B_A \cap B_{A^{-1}}$ is strictly larger than $A'(A)$ whenever $\sigma(A)$ meets the circle of radius r in at least two points, for some $r > 0$. (Take A a unitary operator, not a multiple of the identity.) On the other hand, P.G.Roth [7] gave an example of an operator A such that $\sigma(A) = \{-1,1\}$ but, nevertheless, $B_A = B'(A)$.

If A is decomposable (in the sense of C.Foiaş [2]) and $\sigma(A)$ is included in an increasing arc $\Gamma(A)$, then for each $r\, e^{i\phi(r)} \in \Gamma(A)$, the maximal spectral subspace

$$M_r = \{f \in H: \sigma_A(f) \subset \Gamma(A) \cap \{\lambda: |\lambda| \leqslant r\}\}$$

is hyperinvariant under A and admits the following characterization

$$M_r = \{f \in H: \text{for each } \varepsilon > 0, \text{ there exists } C(f,\varepsilon) \text{ such that } \|A^n f\| \leqslant C(f,\varepsilon)\,[r(1+\varepsilon)]^n \text{ for all } n \geqslant 1\}.$$

Suppose that $\|A^n X A^{-n}\| \leqslant C(X)$ for all $n \in \mathbf{Z}$ and $f \in M_r$; then for each $\varepsilon > 0$,

$$\|A^n X f\| = \|(A^n X A^{-n}) A^n f\| \leqslant C(X)\, \|A^n f\| \leqslant C(X) X(f,\varepsilon)\, [r(1+\varepsilon)]^n$$

$(n = 1,2,\ldots)$, whence it readily follows that $XM_r \subset M_r$.

If $N_r = \{f \in H: \sigma_A(T) \subset \Gamma(A) \cap \{\lambda: |\lambda| \leqslant r\}\}$, then a similar argument shows that $XN_r \subset N_r$.

Thus, we have the following

PROPOSITION 4. *If* A *is decomposable and* $\sigma(A)$ *is included in an increasing arc* $\Gamma(A)$, *then*

$$B_A \cap B_{A^{-1}} \subset \{X \in L(H): XM_r \subset M_r,\ XN_r \subset N_r \text{ for all } r > 0\}.$$

What is the answer to Problem 3 for the case when A is a decomposable operator?.

Recently, D.A.Herrero and H.-W.Kim [6] proved that there is a T in $L(H)$ such that

$$(\operatorname{ran} \delta_T)^- \supset \{\lambda I + K: \lambda \in \mathbf{C},\ K \in K(H)\}.$$

Williams' result [11] play an important role in the proof. C.M.Pearcy observed that if A, $X \in L(H)$, A invertible, $X \neq 0$, and

$$\|A^n X A^{-n}\| \leqslant C(X) \text{ for } n \geqslant 1 \quad \Rightarrow \quad \inf_{n \in \mathbf{Z}} \|A^n X A^{-n}\| > 0\ ,$$

then the proofs given in [6] can be strongly simplified.

Unfortunately, this is not the case.

EXAMPLE 5. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis of H. Define $Be_n = (1/2)e_{n+1}$ $(n \leq 0)$, $Be_n = 2e_{n+1}$ $(n > 0)$, and let 1 denote the identity operator on $\mathbb{C}^1$ (orthonormal basis $\{e\}$).

Let $A = B \oplus 1$, and let R be the rank-one nilpotent operator defined by $Re_0 = e$, $R|\{e_0\}^{\perp} = 0$. For $n \geq 1$, we have

$$\|A^n RA^{-n}\| = \|(B\oplus 1)^n R(B\oplus 1)^{-n}e_n\| = 2^{-n}\|(B\oplus 1)^n Re_0\| =$$
$$= 2^{-n}\|(B\oplus 1)^n e\| = 2^{-n}$$

and

$$\|A^{-n}RA^n\| = \|(B\oplus 1)^{-n}R(B\oplus 1)^n e_{-n}\| = 2^{-n}\|(B\oplus 1)^{-n}Re_0\| =$$
$$= 2^{-n}\|(B\oplus 1)^{-n}e\| = 2^{-n}.$$

Thus, $\sup_{n\in\mathbb{Z}} \|A^n RA^{-n}\| < \infty$ and $\|A^n RA^{-n}\| \to 0$ $(|n| \to \infty)$.

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Arizona State University
Tempe, AZ 85287, USA.

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