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A CONVERSE OF THE ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY

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SUMMARY. In this note we prove: Let (x_ν) $(\nu = 1,2,\ldots)$ be a sequence of positive real numbers. If (x_ν) $(\nu = 0,1,\ldots)$ with $x_0 = 0$ is concave, then

$$\frac{1}{n}\sum_{\nu=1}^{n} x_\nu < \frac{e}{2}\prod_{\nu=1}^{n} x_\nu^{1/n}$$

where the constant $e/2$ is best possible.

For the famous inequality between the (unweighted) arithmetic and geometric means of positive real numbers $x_1,\ldots,x_n$, i.e.

$$G_n = \prod_{\nu=1}^{n} x_\nu^{1/n} \leqslant \frac{1}{n}\sum_{\nu=1}^{n} x_\nu = A_n$$

with equality holding if and only if $x_1 = \ldots = x_n$, one can find in literature many proofs, extensions, sharpenings and variants [1], [3], [5], [6].

In several interesting papers different authors have published upper bounds for the difference $A_n - G_n$ as well as for the ratio A_n/G_n. The resulting inequalities are called complementary or converse inequalities; see [3].

In 1933 J.Favard [4] discovered a remarkable inequality which is complementary to the inequality between the arithmetic mean and the geometric mean of a positive, integrable function. He proved:

If $f\colon [a,b] \to \mathbf{R}$ is positive, continuous and concave, then

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{e}{2}\exp\left[\frac{1}{b-a}\int_a^b \log f(t)dt\right]$$

where the constant $e/2$ cannot be replaced by a smaller number.

Interesting extensions of this result were given by L.Berwald [2].

It is natural to ask: Does there exist a discrete analogue of Favard's inequality? The following proposition gives an affirmative answer to this question.

THEOREM. *Let* (x_ν) $(\nu = 1,2,\ldots)$ *be a sequence of positive real numbers. If* (x_ν) $(\nu = 0,1,\ldots)$ *with* $x_0 = 0$ *is concave, then*

$$\frac{1}{n}\sum_{\nu=1}^{n} x_\nu < \frac{e}{2}\prod_{\nu=1}^{n} x_\nu^{1/n} \tag{1}$$

where the constant $e/2$ *is best possible.*

Proof. We establish (1) by induction on n.

For n = 1 inequality (1) is obviously true.

Next we assume that (1) holds. Then we obtain

$$\prod_{\nu=1}^{n+1} x_\nu = x_{n+1}\prod_{\nu=1}^{n} x_\nu > x_{n+1}\left(\frac{2}{e}\frac{1}{n}\sum_{\nu=1}^{n} x_\nu\right)^n$$

and we have to show that

$$x_{n+1}\left(\frac{2}{e}\frac{1}{n}\sum_{\nu=1}^{n} x_\nu\right)^n \geq \left(\frac{2}{e}\frac{1}{n+1}\sum_{\nu=1}^{n+1} x_\nu\right)^{n+1}. \tag{2}$$

Because of

$$e > \left(1+\frac{1}{n+1}\right)^{n+1}$$

inequality (2) is proved if we can establish

$$\frac{x_{n+1}}{2}\left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{1}{n}\sum_{\nu=1}^{n} x_\nu\right)^n \geq \left(\frac{1}{n+1}\sum_{\nu=1}^{n+1} x_\nu\right)^{n+1}$$

which can be written as

$$\frac{1}{2}\,\frac{(n+2)^{n+1}}{n^n}\,\frac{x_{n+1}}{\sum_{\nu=1}^{n} x_\nu} \geqslant \Big(1 + \frac{x_{n+1}}{\sum_{\nu=1}^{n} x_\nu}\Big)^{n+1} . \tag{3}$$

Now we prove the inequalities

$$\frac{1}{2n} \leqslant \frac{x_{n+1}}{\sum_{\nu=1}^{n} x_\nu} \leqslant \frac{2}{n} . \tag{4}$$

Since (x_ν) is concave we obtain

$$\sum_{i=0}^{2k-2} (x_{n+1-k+i}+x_{n+3-k+i}) \leqslant 2 \sum_{i=0}^{2k-2} x_{n+2-k+i} \qquad (k = 1,\ldots,n+1)$$

which is equivalent to

$$x_{n+1-k}+x_{n+1+k} \leqslant x_{n+2-k}+x_{n+k}.$$

This means that

$$k \longmapsto x_{n+1-k}+x_{n+1+k} \quad (k = 1,\ldots,n+1)$$

is decreasing and we get

$$x_{n+1-k} \leqslant x_{n+1-k}+x_{n+1+k} \leqslant x_n+x_{n+2} \leqslant 2x_{n+1}$$

which implies

$$\sum_{k=1}^{n} x_{n+1-k} \leqslant \sum_{k=1}^{n} 2x_{n+1} ,$$

i.e.

$$\frac{1}{2n} \leqslant \frac{x_{n+1}}{\sum_{i=1}^{n} x_i} .$$

Next we note that the sequence (x_ν/ν) $(\nu = 1,2,\ldots)$ is decreasing. Indeed, since

$$\frac{x_0+x_2}{2} = \frac{x_2}{2} \leqslant x_1$$

and because of

$$(\nu+1)\ \Big[\frac{x_{\nu+1}}{\nu+1} - \frac{x_\nu}{\nu}\Big] \leqslant (\nu-1)\ \Big[\frac{x_\nu}{\nu} - \frac{x_{\nu-1}}{\nu-1}\Big]$$

the assertion follows immediately by induction.

We prove the second inequality of (4) by induction on n. For n=1 the inequality is obviously valid. From the monotonicity of (x_ν/ν) and from the induction hypothesis we obtain

$$(n+1)x_{n+2} - 2\sum_{\nu=1}^{n+1} x_\nu \leqslant (n+2)x_{n+1} - 2\sum_{\nu=1}^{n+1} x_\nu =$$

$$= nx_{n+1} - 2\sum_{\nu=1}^{n} x_\nu \leqslant 0$$

which completes the proof of double-inequality (4).

We denote by f the function

$$f : [\tfrac{1}{2n},\tfrac{2}{n}] \longrightarrow \mathbf{R},$$

$$f(x) = \frac{1}{2}\,\frac{(n+2)^{n+1}}{n^n}\,x - (1+x)^{n+1} .$$

Because of

$$f''(x) = -n(n+1)(1+x)^{n-1} < 0$$

we conclude that f is concave which implies

$$f(x) \geqslant \min\,\{f(\tfrac{1}{2n}),f(\tfrac{2}{n})\}. \tag{5}$$

We have

$$f(\tfrac{2}{n}) = 0 \tag{6}$$

and we will prove

$$f(\tfrac{1}{2n}) = \frac{1}{4}\left(\frac{2n+1}{2n}\right)^{n+1}\left[\left(\frac{n+2}{n+\frac{1}{2}}\right)^{n+1} - 4\right] \geqslant 0 . \tag{7}$$

In order to establish (7) we show that the sequence

$$L_n = \left(\frac{n+2}{n+\frac{1}{2}}\right)^{n+1} \qquad (n = 1,2,\ldots)$$

is increasing. We define

$$g(x) = (x+1)\log\frac{x+2}{x+\frac{1}{2}}\,,\quad x \geqslant 1.$$

Differentiation yields

$$g'(x) = \log \frac{x+2}{x+\frac{1}{2}} - \frac{3(x+1)}{2(x+\frac{1}{2})(x+2)} .$$

Replacing in

$$\log (1+\frac{1}{y}) > \frac{2}{2y+1} , \quad y > 0 ,$$

(see [6,p.273]) y by $\frac{2x+1}{3}$ we obtain for $x \geqslant 1$:

$$\log \frac{x+2}{x+\frac{1}{2}} > \frac{2}{\frac{2}{3}(2x+1)+1} \geqslant \frac{3(x+1)}{2(x+\frac{1}{2})(x+2)}$$

which yields

$$g'(x) > 0 \quad \text{for} \quad x \geqslant 1.$$

Hence we get for $n \geqslant 1$:

$$L_n \geqslant L_1 = 4$$

and therefore

$$\delta(\frac{1}{2n}) \geqslant 0.$$

From (5), (6) and (7) we conclude

$$\delta(x) \geqslant 0 \quad \text{for all} \quad x \in [\frac{1}{2n}, \frac{2}{n}] .$$

This proves inequality (3).

Finally we have to show that in (1) the constant $e/2$ cannot be replaced by a smaller number.

We assume that the inequality

$$\frac{1}{n} \sum_{\nu=1}^{n} x_\nu < c \prod_{\nu=1}^{n} x_\nu^{1/n} \tag{8}$$

holds for every sequence (x_ν) ($\nu = 1,2,\ldots$) of positive real numbers such that (x_ν) ($\nu = 0,1,\ldots$) with $x_0 = 0$ is concave.

Let $x_\nu = \nu/n$ ($\nu = 0,1,\ldots;n \in \mathbf{N}$). Then we obtain from (8):

$$\frac{n+1}{2n} < c \frac{(n!)^{1/n}}{n} \quad \text{or} \quad \frac{n+1}{2(n!)^{1/n}} < c.$$

If n tends to ∞ we get:

$$e/2 \leq c.$$

Hence, the constant $e/2$ is best possible in (1).

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