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A NOTE ON THE SPACE OF GEODESICS

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ABSTRACT. Let M be a complete C^∞-Riemannian manifold of dimension $n \geqslant 2$, simply connected and without focal points. In this paper we prove that the manifold of oriented geodesics $\vec{G}$ of M is defined, and it is naturally diffeomorphic to the tangent bundle TS^{n-1} of the n-1-dimensional standard sphere S^{n-1}. If $d\vec{G}^{n-1}$ denotes the standard volume of $\vec{G}$, we also get the global expression of $d\vec{G}^{n-1}$ on TS^{n-1}. Hence concerning geodesics and in contrast with the horosphere case (see [5]), Integral Geometry or Geometric Probability may be applied to this class of Riemannian manifolds, in particular to Hadamard manifolds.

INTRODUCTION

Throughout let M be a complete C^∞-Riemannian manifold of dimension $n \geqslant 2$ and inner product $\langle , \rangle$. Let TM and T_1M denote respectively the tangent bundle and the sphere bundle of M, and π be the projection map onto M in either case.

We also denote the fibre of TM and T_1M over $p \in M$ respectively by M_p and S_p. Let $\vec{G}$ be the set of leaves of the geodesic spray restricted to T_1M. Since an oriented geodesic regarded as a 1-dimensional (immersed) oriented submanifold of M can be identified with a point of $\vec{G}$ and viceversa, we call $\vec{G}$ the set of oriented geodesics of M. Let G be the set of non-oriented geodesics, where a point of G is obtained identifying two points of $\vec{G}$ in the obvious way. If the geodesic spray restricted to T_1M defines a regular foliation in the sense of Palais (see [6]), then $\vec{G}$ becomes a differentiable manifold of dimension 2n-2. We shall say in this case that the manifold of geodesics $\vec{G}$ of M is defined.

DEFINITION. We shall say that M is G-measurable if the manifold $\vec{G}$ is defined and $\vec{G}$ is Hausdorff; hence G is also a manifold of dimension 2n-2.

The reason of this definition is the following:

Let $\Gamma: T_1M \to \vec{G}$ and $\xi: \vec{G} \to G$ be the projection mappings and ω the symplectic form restricted to T_1M; then one gets (see [4])

PROPOSITION. *If* M *is* G-*measurable, there exists a unique* 2-*form* $d\vec{G}$ *on* $\vec{G}$ *which satisfies* $\Gamma^*(d\vec{G}) = \omega$. *Let* $d\vec{G}^{n-1}$ *be the* 2n-2-*form on* $\vec{G}$ *defined by* $d\vec{G}^{n-1} = d\vec{G} \wedge \ldots \wedge d\vec{G}$, *then* $d\vec{G}^{n-1}$ *is a volume on* $\vec{G}$ *and it is invariant under the group of isometries of* M *acting on* $\vec{G}$. *Moreover, a non-vanishing absolute* 2n-2-*form* $|dG^{n-1}|$ *is defined on* G *with the same invariance property.*

If n is odd, then $|dG^{n-1}|$ turns into a form dG^{n-1} which satisfies $\xi^*(dG^{n-1}) = d\vec{G}^{n-1}$; and consequently G is orientable in this case.

REMARK. Compare the results of the above proposition with those of Besse (see [1]) obtained under more restrictive conditions on the geodesic flow. Also the concept of measure of geodesics described by Santaló (see [7]).

The main result of this paper is to prove:

THEOREM. *Let* M *be a complete* C^∞-*Riemannian manifold of dimension* $n \geqslant 2$, *simply connected and without focal points. Then* M *is G-measurable and $\vec{G}$ is naturally diffeomorphic to* TS^{n-1}, *where* S^{n-1} *is the standard* n-1 *dimensional unit sphere.*

REMARK. From the theorem follows that in order to measure sets of oriented geodesics of this class of Riemannian manifolds the manifold TS^{n-1} can be used as a model. The corresponding volume $d\vec{G}^{n-1}$ on TS^{n-1} will be obtained as a corollary of our main result.

This work is divided in two sections: the first one sketches under the heading "Preliminaries" some basic materials which are needed, whereas the second one is intended to prove our result.

Before passing to the first section

REMARK. The following questions remain unanswered:

1. Is the Hausdorff condition in the above definition superflous?. (Indeed it is, if M is compact).
2. If M is G-measurable, the 2k-dimensional submanifolds of $\vec{G}$ (k = 1,...,n-1) can be measured with the non-trivial form $d\vec{G}^k$. Consequently, assume there exists a 2k-1-form α on $\vec{G}$ with k = 1,...,n-1 such that α is invariant under the group of isometries acting on $\vec{G}$. What can be said about α? Is $\alpha = 0$?.

1. PRELIMINARIES

The reader is referred to ([1], Ch.1) and [3] for basic facts not defined here.

For each $v \in TM$, we denote the geodesic with initial velocity v by c_v and the tangent vector to c_v at "t" by $c_v'(t)$.

Thus the geodesic flow $\phi: TM \times R \to TM$ is defined by $\phi(v,t) = c_v'(t)$.

Let $K: T(TM) \to TM$ be the connection map induced by <,>, and π_* the differential map of π. It follows that $\pi_* \times K: T(TM) \to TM \times TM$

defined by $\pi_* \times K(b) = (\pi_*(b), K(b))$ is an isomorphism restricted to each fibre. In terms of this map, the geodesic spray S is defined for any $v \in TM$ by $S(v) = (\pi_* \times K)^{-1}\ (v, 0_{\pi(v)})$. Let g be the Sasaki metric on TM, which is defined for any pair of vector fields X, Y on TM by $g(X,Y) = \langle \pi_* X, \pi_* Y\rangle + \langle KX, KY\rangle$.

Orienting TM by its natural charts, the volume on TM induced by g will be denoted by dTM. For any k-form α on TM $(k > 0)$ and any vector field X on TM let $C_X \alpha$ be the k-1-form on TM defined by $C_X\alpha(X_1,\ldots,X_{k-1}) = \alpha(X, X_1,\ldots,X_{k-1})$.

Let N be the vector field on TM defined by $N(v) = (\pi_* \times K)^{-1}\ (0_{\pi(v)}, v)$; then $dT_1M = C_N\ dTM$ induces a volume on T_1M.

Let θ be the 1-form on TM defined by $\theta(X) = g(S,X)$, and $\omega = -d\theta$ the symplectic form on TM; then

$$\omega(X,Y) = \langle \pi_* X, KY\rangle - \langle KX, \pi_* Y\rangle.$$

If $\omega^n = \omega \wedge \ldots \wedge \omega$ then $\omega^n = n!.(-1)^{(n/2)}.dTM$; and since $C_N\omega = -\theta$, one gets

(1) $$\omega^{n-1} \wedge \theta = k_n . dT_1M$$

where $k_n = (n-1)!.(-1)^{(n/2)+1}$.

If S and ω are restricted to T_1M and X is a vector field on T_1M, it follows that

(2) $$\theta(S) = 1 \quad \text{and} \quad \omega(S,X) = 0.$$

Thus the following equality holds on T_1M

(3) $$\omega^{n-1} = k_n . C_S\ dT_1M.$$

DEFINITION. Let $f: A\times R \to M$ be a smooth map, where A is an open set of R^m $(m > 0)$. We shall call f a family of geodesics depending on m-parameters if for any $x \in A$ the curve

$f(x,\ ): R \to M$ which maps $t \to f(x,t)$ is a unit speed geodesic.

REMARK. Let $f'(x,t)$ be the tangent vector to $f(x,\ )$ at "t", and $f'_0(x) = f'(x,0)$. Then f induces mappings $\vec{G}f: A \to \vec{G}$ and $Gf: A \to G$ defined by $\vec{G}f(x) = \Gamma(f'_0(x))$ and $Gf = \xi \circ \vec{G}f$. Clearly if M is G-measurable, then $\vec{G}f$ and Gf are smooth mappings.

DEFINITION. Assuming that the manifold $\vec{G}$ is defined, if f depends on 2n-2 parameters, we shall say that f induces a coordinate system on $\vec{G}$ if the map $\vec{G}f: A \to \vec{G}f(A)$ is a chart for $\vec{G}$.

The following three lemmas will be needed later on. The first one follows easily from equalities (1) and (2) and the fact that ω is ϕ_t-invariant. The second follows also from this invariant fact, and the third one can be checked locally.

LEMMA 1.1. *Let* $f: A \times R \to M$ *be a family of geodesics depending on* 2n-2 *parameters. Let* $dx = dx_1 \wedge \dots \wedge dx_{2n-2}$ *and* θ_f *be the function such that* $(f')^*(dT_1M) = \theta_f . dx \wedge dt$. *Then* θ_f *does not depend on* t *and* $(f'_0)^*(\omega^{n-1}) = k_n . \theta_f . dx$.

LEMMA 1.2. *Assume that the manifold* $\vec{G}$ *is defined, and let* $f: A \times R \to M$ *be a family of geodesics depending on* 2n-2 *parameters such that* $\vec{G}f: A \to \vec{G}f(A)$ *is a bijective map. Suppose that for any* $x \in A$ *the differential map of* $f': A \times R \to T_1M$ *at* (x,0) *is an isomorphism. Then* f *induces a coordinate system on* $\vec{G}$.

LEMMA 1.3. *Identify* T_1S^{n-1} *with the set of pairs* $(v,u) \in S^{n-1} \times S^{n-1}$ *such that* $v \perp u$. *Let* $h: T_1S^{n-1} \times (0,+\infty) \to TS^{n-1}$ *be the map defined by* $h(v,u,t) = (u,t.v)$. *Then* $h^*(dTS^{n-1}) =$
$= (-t)^{n-2} . dT_1S^{n-1} \wedge dt$.

2. THE NO FOCAL POINT CASE

Here we assume that M is simply connected and without focal points. We recall that M has no focal points if for any $v \in T_1M$ and any Jacobi field Y along c_v such that $Y(0) = 0$ and $Y'(0) \neq 0$, $\langle Y'(t),Y(t)\rangle > 0$ for any $t > 0$ where Y' denotes the covariant derivative with respect to c_v'. For a fixed point $p \in M$, let us identify TS_p with the set of pairs $(u,w) \in S_p \times M_p$ such that $u \perp w$; and let $\tau: TS_p \to T_1M$ be the map defined by $\tau(u,w) = P_w(u)$, where $P_w: M_p \to M_{c_w(1)}$ is the parallel translation along c_w from p to $c_w(1)$.
For any $v \in S_p$, let $v^\perp$ be the orthogonal subspace of M_p to v; and for any $t > 0$, define $D_{v,t}: v^\perp \to v^\perp$ by $D_{v,t}(u) = P_{t.v}^{-1}(Y_u(t))$ where Y_u is the Jacobi field along c_v which satisfies $Y_u(0) = 0$ and $Y_u'(0) = u$. Let $Y(v,u,t,-)$ be the Jacobi field along c_v such that $Y(v,u,t,0) = 0$ and $Y(v,u,t,t) = P_{t.v}(u)$.
Identifying $(S_p)_u$ with $v^\perp$ we define $\mu_p: TS_p \times (0,+\infty) \to R$ by

(4) $$\mu_p(v,u,t) = \langle Y'(v,u,t,t),P_{t.v}(u)\rangle \,.\, \det D_{v,t}.$$

Since M has no focal points, it follows that $\mu_p(v,u,t) > 0$ if $u \neq 0$. Moreover, since $\frac{1}{t^{n-1}} \,.\, \det D_{v,t} \to 1$ as $t \to 0$ and $t.\langle Y'(v,u,t,t),P_{t.v}(u)\rangle \to |u|^2$ as $t \to 0$, we have

(5) $$\lim_{t\to 0} \frac{1}{t^{n-2}} \,.\, \mu_p(v,u,t) = 1 \quad \text{if} \quad (v,u,t) \in T_1S_p \times (0,+\infty).$$

PROPOSITION 2.1. *Let* $P: TS_p \times (0,+\infty) \to TM$ *be the map defined by* $P(v,u,t) = P_{t.v}(u)$. *Then* $P^*(C_S \, d\, T_1M) = (-1)^{n-1}.\mu_p.\, d\, T_1\, S_p \wedge dt$.

Proof. Equality will be shown locally. Let (V,ψ) be a chart for S_p and $w_1,\dots,w_{n-1}$ be the induced coordinate vector fields on V. If $v \in V$, let $f_{ij} = \langle w_i,w_j\rangle$ and $g_p(v) = \det \|f_{ij}(v)\|$.

Let $(TV,T\psi)$ be the induced chart for TS_p with $T\psi = (y_1,\ldots,y_{n-1},\dot{y}_1,\ldots,\dot{y}_{n-1})$; then

$$d\,T_1\,S_p = (-1)^{n-1}.g_p.dy \wedge \Big(\sum_{k=1}^{n-1} (-1)^{k+1}.\dot{y}_k.d\dot{y}^k\Big)$$

where $dy = dy_1 \wedge \ldots \wedge dy_{n-1}$, $d\dot{y}^k = d\dot{y}_1 \wedge \ldots \wedge \widehat{d\dot{y}_k} \wedge \ldots \wedge d\dot{y}_{n-1}$.

The symbol $\wedge$ over $d\dot{y}_k$ indicates that $d\dot{y}_k$ is omitted.

Applying the Gram-Schmidt process to $w_1,\ldots,w_{n-1}$ we get orthonormal fields $e_1,\ldots,e_{n-1}$ on V.

If $w_i(v) = \sum_{j=1}^{n-1} b_{ij}(v).e_i(v)$, let $B(v) = \|b_{ij}(v)\|$. For any $v \in V$, let $Y_i(v,-)$ be the Jacobi field along c_v such that $Y_i(v,0) = 0$ and $Y_i'(v,0) = e_i(v)$. If $e_i(v,t) = P_{t.v}(e_i(v))$ and

$$Y_i(v,t) = \sum_{j=1}^{n-1} a_{ij}(v,t)\,.\,e_j(v,t),\ \text{let } A(v,t) = \|a_{ij}(v,t)\|.$$

Thus, $\det D_{v,t} = \det A(v,t)$.

Let $\exp\colon M_p \to M$ be the exponential map and for any $\alpha \in D = \psi(V)$ and $t > 0$ let us define $\varphi^{-1}(\alpha,t) = \exp(\psi^{-1}(\alpha).t)$. If $U = \varphi^{-1}(D\times(0,+\infty))$, let $(TU,T\varphi)$ be the induced chart for TM with $T\varphi = (x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n)$. Then

$$dT_1M = (-1)^n.g.dx \wedge \Big(\sum_{k=1}^{n} (-1)^{k+1}.\dot{x}_k.d\dot{x}^k\Big)$$ where

$dx = dx_1 \wedge \ldots \wedge dx_n$, $d\dot{x}^k = d\dot{x}_1 \wedge \ldots \wedge \widehat{d\dot{x}_k} \wedge \ldots \wedge d\dot{x}_n$; and the function g satisfies $g(P(v,u,t)) = g_p(v).\det^2 D_{v,t}$ if $P(v,u,t) \in TV$. An easy check shows now that for any $(v,u,t) \in TV\times(0,+\infty)$

$$x_i \circ P(v,u,t) = y_i(v) \text{ if } i = 1,\ldots,n-1,\quad x_n \circ P(v,u,t) = t,$$

$$\dot{x}_n \circ P(v,u,t) = 0\ ;\ \text{and}\ (\dot{x}_i \circ P(v,u,t),\ldots,\dot{x}_{n-1} \circ P(v,u,t)) =$$

$$= (\dot{y}_1(u),\ldots,\dot{y}_{n-1}(u)).B(v).A^{-1}(v,t).B^{-1}(v).$$

Moreover, if a: TU $\to$ R is the function defined by $a(w) = S(w)(\dot{x}_n)$ then $a(P(v,u,t)) = \langle Y'(v,u,t,t), P_{t.v}(u)\rangle$. Equality on $TU\times(0,+\infty)$ follows now, using the properties of the contraction operator C_S with respect to the exterior product.

From the previous proposition and equality (3) we get

COROLLARY 2.2. Let P: $T_1S_p\times(0,+\infty) \to T_1M$ defined by $P(v,u,t) = P_{t.v}(u)$.
Then $P^*(\omega^{n-1}) = k_n.(-1)^{n-1}.\mu_p.dT_1S_p \wedge dt$.

From equality (5), lemma 1.3. and the previous corollary follows

COROLLARY 2.3. The function $\tau: TS_p \to T_1M$ satisfies $\tau^*(\omega^{n-1}) = \theta_p.dTS_p$, where $\theta_p(u,0) = -k_n$ and $\theta_p(u,w) = \frac{-k_n}{t^{n-2}}.\mu_p(v,u,t)$ if $|w| = t > 0$ and $v = t^{-1}.w$.

REMARK. Since M is simply connected and without focal points it follows (see [2]) that for any geodesic σ of M (where σ is regarded as a 1-dimensional submanifold of M) and for any point $q \in M$ not lying on σ, there exists a unique geodesic through q which intersects σ perpendicularly. Due to this fact, for the fixed point "p", let us define the family of geodesics depending on 2n-2 parameters f: $TS_p\times R \to M$ by $f(u,w,t) = \pi \circ \phi(\tau(u,w),t)$.

Hence $f': TS_p\times R \to T_1M$ and $\vec{G}f: TS_p \to \vec{G}$ are bijective mappings. Since S^{n-1} is isometric to S_p, we can state our main result as follows

THEOREM 2.4. *The manifold* $\vec{G}$ *of* M *is defined, and* $\vec{G}f: TS_p \to \vec{G}$ *is a diffeomorphism, hence* M *is* G-*measurable.*

Proof. Since $f'(u,w,t) = \phi(\tau(u,w),t)$ then $f'_0 = \tau$; hence by lemma 1.1 and the previous corollary we get

$(f')^*(dT_1M) = k_n^{-1}.\theta_p dTS_p \wedge dt.$

Since f' is a bijective map and $k_n^{-1}.\theta_p < 0$, it follows that f' is a diffeomorphism; consequently the geodesic spray restricted to T_1M defines a regular foliation. Hence the manifold $\vec{G}$ is defined.

Since $\vec{G}f$ is a bijective map, by virtue of lemma 1.2 we get that $\vec{G}f$ is a diffeomorphism.

COROLLARY 2.5. The volume $d\vec{G}^{n-1}$ on $\vec{G}$ is represented on TS_p (via $\vec{G}f$) by the volume $\theta_p.dTS_p$.

Proof. Since $\vec{G}f = \Gamma \circ f_0'$ one gets

$(\vec{G}f)^*(d\vec{G}^{n-1}) = (f_0')^* \circ \Gamma^*(d\vec{G}^{n-1}) = (f_0')^*(\omega^{n-1}) = \tau^*(\omega^{n-1}) =$

$= \theta_p.dTS_p.$

EXAMPLE. If M has constant sectional curvature $K = -r^2$ $(r \geqslant 0)$ then in the Euclidean case $(r = 0)$ we get $\theta_p = -k_n$. In the hyperbolic case $(r > 0)$ we get $\theta_p(u,0) = -k_n$ and

$\theta_p(u,w) = -k_n.Ch(r.t).(\frac{Sh(r.t)}{r.t})^{n-2}$ if $|w| = t > 0$.

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