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STRONGLY PERFECTLY PROPER EQUILIBRIUM

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ABSTRACT. Perfect and proper equilibrium points have been refined very recently by García Jurado and Prada Sánchez who introduced the notions of strongly proper and perfectly proper equilibria. Here we further refine such concepts to strongly perfectly proper equilibrium points, we prove their existence and that they constitute a proper refinement.

I. INTRODUCTION

One of the most important solutions introduced in non-cooperative normal games is the concept of equilibrium point due to Nash [6]. Generally this point is not unique, for this reason it is convenient to establish criteria selecting the most convenient and intuitive ones.

In this way Selten [1975] introduced the concept of perfect equilibrium to eliminate solutions which are not stable against any arbitrarily slight perturbation of the game strategies. Only those equilibria which are self-enforcing under some arbitrarily slight player mistakes are acceptable.

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Myerson [1978] went further. He assumed that the players err "rationally" in the sense that their errors have probabilities that fall as the cost of the error rises. Myerson's concept of proper equilibrium eliminates those Nash equilibria that are not stable under any arbitrarily slight rational player mistake.

Recently García Jurado [1988] went further and he considered not only that players will tend to err with more probability towards that which cost less. But that besides it will tend to err more those players having a lower the cost. He introduced the concept of perfectly proper equilibrium point and proved its existence.

In another note García Jurado and Prada Sánchez [3] have introduced the concept of strongly proper equilibrium to rule out Nash equilibria that are not stable under any arbitrarily slight rational players error. Meaning rational is assuming that errors with equal cost have equal probabilities.

In this paper we introduce the concept of strongly perfectly proper equilibrium point, by short spp equilibrium point joining together the last two aspects.

2. CONCEPTS

In this section we introduce our notation and define the equilibrium concepts introduced by Selten, Myerson, García Jurado and García Jurado - Prada Sánchez.

Let Γ be a n-person game in normal form, i.e.

$$\Gamma = (\Phi_1,\ldots,\Phi_n \; ; H_1,\ldots,H_n)$$

where Φ_i is the finite set of pure strategies of player i and H_i defined from $\Phi_1 \times \ldots \times \Phi_n$ to the reals R is his payoff function. Let S_i be the set of mixed strategies of player i, i.e.

$$S_i = \{s_i \in R^{|\Phi_i|} : \sum_{\phi_i \varepsilon \Phi_i} s_i(\phi_i) = 1 \quad s_i(\phi_i) \geqslant 0 \quad \forall \; \phi_i \in \Phi_i\}.$$

An $s_i \in S_i$ is completely mixed if $s_i(\phi_i) > 0 \quad \forall \phi_i \in \Phi_i$.

Each $s = (s_1,\ldots,s_n) \in S = S_1 \times \ldots \times S_n$ is termed a combination of strategies, and for $\bar{s}_i \in S_i$ the combination of strategies

$$(s_1,\ldots,s_{i-1},\bar{s}_i,s_{i+1},\ldots,s_n)$$

is denoted by $s\backslash\bar{s}_i$. An $s = (s_1,\ldots,s_n) \in S$ is completely mixed if all its components s_i are. We recall that the set of pure strategies Φ_i can be treated as a subset of S_i by identifying each ϕ_i with that $s_i \in S_i$ for which $s_i(\phi_i) = 1$ and $s_i(\phi_i') = 0$ for all $\phi_i' \in \Phi_i \setminus \{\phi_i\}$.

The extension of H_i to S is defined by

$$H_i(s) = H_i(s_1,\ldots,s_n) = \sum_{(\phi_1,\ldots,\phi_n) \in \Phi_1 \times \ldots \times \Phi_n} H_i(\phi_1,,,\phi_n) \prod_{j=1}^{n} s_j(\phi_j)$$

and s_i, $\bar{s}_i \in S_i$ are said to be payoff-equivalent for player i if

$$H_i(s\backslash s_i) = H_i(s\backslash\bar{s}_i) \quad \forall \quad s \in S.$$

$\phi_i \in \Phi_i$ is a best pure response to $s \in S$ for player $i \in N$ if

$$H_i(s\backslash\bar{\phi}_i) = \max_{\phi_i \varepsilon \Phi_i} H_i(s\backslash\phi_i).$$

The set of all player i's best pure responses to s will be denoted by $B_i(s)$. $\bar{s}_i \in S_i$ is a best response to $s \in S$ for player i if for all $s_i \in S_i$

$$H_i(s\backslash\bar{s}_i) \geqslant H_i(s\backslash s_i).$$

$\bar{s} = (\bar{s}_1,\ldots,\bar{s}_n) \in S$ is a best response to $s \in S$ if for all i, $\bar{s}_i$ is a best response to s for player i. $s \in S$ is a Nash equilibrium if it is a best response to itself.

An $s = (s_1,\ldots,s_n) \in S$ is an ε - perfect equilibrium if it is completely mixed and, $\forall \ \phi_i$, $\forall \ \bar{\phi}_i$ and $\forall$ i

$$H_i(s\backslash\phi_i) < H_i(s\backslash\overline{\phi}_i) \Rightarrow s_i(\phi_i) \leqslant \varepsilon.$$

An $s = (s_1,\ldots,s_n) \in S$ is a perfect equilibrium if there exists a pair of sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty} = \{(s_1^k,\ldots,s_n^k)\}_{k=1}^{\infty}$ such that:

a) $\varepsilon_k > 0 \ \forall \ k, \ \lim_{k\to\infty} \varepsilon_k = 0$

b) s_k is an ε_k - perfect equilibrium $\forall$ k.

c) $\lim_{k\to\infty} s_i^k(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \Phi_i$ and $\forall$ i.

An $s = (s_1,\ldots,s_n) \in S$ is an ε - proper equilibrium if it is completely mixed and, $\forall \ \phi_i, \ \forall \ \overline{\phi}_i \in \Phi_i$ and $\forall$ i

$$H_i(s\backslash\phi_i) < H_i(s\backslash\overline{\phi}_i) \Rightarrow s_i(\phi_i) \leqslant \varepsilon \ s_i(\overline{\phi}_i).$$

An $s = (s_1,\ldots,s_n) \in S$ is a proper equilibrium if there exists a pair of sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty} = \{(s_1^k,\ldots,s_n^k)\}_{k=1}^{\infty}$ such that

a) $\varepsilon_k > 0 \ \forall \ k, \ \lim_{k\to\infty} \varepsilon_k = 0$

b) s_k is an ε_k - proper equilibrium $\forall$ k.

c) $\lim_{k\to\infty} s_i^k(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \Phi_i$ and $\forall$ i.

A strongly ε-proper equilibrium is a point $s = (s_1,\ldots,s_n) \in S$ if:

a) It is ε - proper equilibrium

b) $\forall \ \phi_i, \ \overline{\phi}_i \in \Phi_i \backslash B_i(s)$ and $\forall$ i, $s_i(\phi_i) = s_i(\overline{\phi}_i)$ if ϕ_i and $\overline{\phi}_i$ are payoff-equivalent for player i.

An $s = (s_1,\ldots,s_n) \in S$ is a strongly proper equilibrium point if there exists a pair of sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty} = \{(s_1^k,\ldots,s_n^k)\}_{k=1}^{\infty}$ such that:

a) $\varepsilon_k > 0 \ \forall \ k, \ \lim_{k\to\infty} \varepsilon_k = 0$

b) s_k is a strongly ε_k - proper equilibrium $\forall$ k

c) $\lim_{k\to\infty} s_i^k(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \Phi_i$ and $\forall$ i.

García Jurado and Prada Sánchez in [3] have proved the existence of strongly proper equilibrium point which is a proper refinement of the concept of proper equilibrium point. They gave the following example

	β_1		β_2		β_3	
α_1	2	2	1	1	1	1
α_2	2	2	0	1	2	1
α_3	1	2	1	300	1	2
α_4	1	2	1	1	1	3

The points (α_1,β_1) and (α_2,β_1) are proper equilibrium points but only the point (α_1,β_1) is strongly proper.

A point $s = (s_1,\dots,s_n)$ is an ε - perfectly proper equilibrium point if and only if:

a) s is completely mixed

b) If $H_i(s\backslash\phi_i)-H_i(s\backslash\overline{\phi}_i) < H_j(s\backslash\phi_j)-H_j(s\backslash\overline{\phi}_j)$ with $\phi_i \in B_i(s)$, $\phi_j \in B_j(s)$ then $s_j(\overline{\phi}_j) \leq \varepsilon \ s_i(\overline{\phi}_i) \ \forall \ \overline{\phi}_i \in \Phi_i, \ \forall \ \overline{\phi}_j \in \Phi_j$ $\forall$ i, j $\in \{1,\dots,n\}$.

It is said that a point $s = (s_1,\dots,s_n) \in S$ is a perfectly proper equilibrium point if and only if there exists a pair of sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty} = \{(s_1^k,\dots,s_n^k)\}_{k=1}^{\infty}$ such that

a) $\varepsilon_k > 0 \ \forall \ k, \ \lim_{k\to\infty} \varepsilon_k = 0$

b) s_k is ε_k - perfectly proper $\forall$ k

c) $\lim\limits_{k\to\infty} s_i^k(\phi_i) = s_i(\phi_i) \quad \forall\ \phi_i \in \Phi_i \quad \forall\ i \in \{1,\dots,n\}$.

García Jurado in [2] has introduced a concept properly refining the concept of proper equilibrium point.

In his doctoral thesis Van Damme [1983] shows the fact that enlarging dominated strategies the set of proper equilibrium might enlarge too. He used the following example

Γ_1

	α_1^2	α_2^2
α_1^1	1 1 1	0 0 1
α_2^1	0 0 1	0 0 1

It is clear that the unique proper equilibrium point in Γ_1 is (α_1^1,α_1^2). Consider the game Γ_2 which is obtained enlarging a strategy strictly dominated for the third player in the game Γ_1,

α_1^3

	α_1^2	α_2^2
α_1^1	1 1 1	0 0 1
α_2^1	0 0 1	0 0 1

α_2^3

	α_1^2	α_2^2
α_1^1	0 0 0	0 0 0
α_2^1	0 0 0	1 1 0

In Γ_2 the points $(\alpha_1^1,\alpha_1^2,\alpha_1^3)$ and $(\alpha_2^1,\alpha_2^2,\alpha_1^3)$ are proper equilibrium points. García Jurado in [6] proved that the point $(\alpha_2^1,\alpha_2^2,\alpha_1^3)$ is not perfectly proper.

3. STRONGLY PERFECTLY PROPER EQUILIBRIUM POINT

As we mentioned in the Introduction we introduce a further concept which will result to be a proper refinement of the concept of perfectly proper equilibrium point.

We say that a point $s = (s_1,\dots,s_n) \in S$ is ε - strongly perfectly proper equilibrium point or briefly ε - s.p.p. equilibrium point if:

a) s is completely mixed

b) if $H_i(s\backslash\phi_i)-H_i(s\backslash\overline{\phi}_i) < H_j(s\backslash\phi_j)-H_j(s\backslash\overline{\phi}_j)$ with $\phi_i \in B_i(s)$ and $\phi_j \in B_j(s)$ then $s_j(\overline{\phi}_j) \leqslant \varepsilon\, s_i(\overline{\phi}_i)$ $\forall\, \overline{\phi}_i \in \Phi_i$, $\forall\, \overline{\phi}_j \in \Phi_j$ $\forall\, i,j \in \{1,\dots,n\}$

c) $\forall\, \phi_i, \overline{\phi}_i \in \Phi_i\backslash B_i(s)$ and $\forall\, i$, $s_i(\phi_i) = s_i(\overline{\phi}_i)$ if ϕ_i and $\overline{\phi}_i$ are payoff-equivalent for player i.

A point $s = (s_1,\dots,s_n) \in S$ is called to be strongly perfectly proper equilibrium point or briefly s.p.p. equilibrium point if there exists a pair of sequences
$\{\varepsilon_k\}_{k=1}^{\infty}$, $\{(s_1^k,\dots,s_n^k)\}_{k=1}^{\infty} = \{s_k\}_{k=1}^{\infty}$ such that

a) $\forall\, k\ \varepsilon_k > 0;\ \lim\limits_{k\to\infty} \varepsilon_k = 0$

b) $\forall\, k\ s_k$ is a ε - s.p.p. equilibrium point

c) $\lim\limits_{k\to\infty} s_i^k(\phi_i) = s_i(\phi_i)$ $\forall\ \phi_i \in \Phi_i$, $\forall\ i \in \{1,\dots,n\}$.

It is clear that a s.p.p. equilibrium point is a perfectly proper equilibrium point. The inverse is not true in general. Consider the example

	β_1	β_2	β_3
α_1	2 2	1 1	1 1
α_2	2 2	0 1	2 1
α_3	1 2	1 301	1 2
α_4	1 2	1 1	1 3
α_5	2 2	1 1	1 1

Γ

It is easy to check that the points $(\mu\alpha_1+(1-\mu)\alpha_5;\beta_1)$, $\mu \in [0,1]$ and (α_2,β_1) are perfectly proper equilibrium points. Consider first the point (α_2,β_1). Taking the sequences

$$\varepsilon_k = \frac{1}{k+2}$$

$$s_1^k(\alpha_1) = s_1^k(\alpha_5) = \frac{1}{2(k+2)} \qquad k = 4,5,\ldots$$

$$s_1^k(\alpha_2) = 1 - \frac{300(k+2)^2+151}{300(k+2)^3} \; ; \quad s_1^k(\alpha_3) = \frac{1}{300(k+2)^3} \; ; \; s_1^k(\alpha_4) = \frac{1}{2(k+2)^3}$$

$$s_2^k(\beta_1) = 1 - \frac{k+3}{2(k+2)^3} \; ; \; s_2^k(\beta_2) = \frac{1}{2(k+2)^3} \; ; \; s_2^k(\beta_3) = \frac{1}{2(k+2)^2}$$

then we have

$$H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_3) = 1 - \frac{1}{(k+2)^3}$$

$$H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_1) = \frac{k+1}{2(k+2)^3}$$

and

$$H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_5) = H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_1) < H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_3)$$

$$s_1^k(\alpha_3) \leqslant \varepsilon_k s_1^k(\alpha_1) \quad \text{and} \quad s_1^k(\alpha_3) \leqslant \varepsilon_k s_1^k(\alpha_5)$$

hold true. The same for α_1 and α_4. For the second player

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2) = 1 - \frac{1}{(k+2)^3}$$

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3) = 1 - \frac{301}{300(k+2)^3}$$

and

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2)$$

$$s_2^k(\beta_2) \leqslant \varepsilon_k s_2^k(\beta_3)$$

holds true.

Finally

$$H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_5) = H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_1) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2)$$

$$s_2^k(\beta_2) \leqslant \varepsilon_k s_1^k(\alpha_1)$$

$$s_2^k(\beta_2) \leqslant \varepsilon_k s_1^k(\alpha_5)$$

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3) < H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_3) = H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_4)$$

$$s_1^k(\alpha_3) \leqslant \varepsilon_k s_2^k(\beta_3)$$

$$s_1^k(\alpha_4) \leqslant \varepsilon_k s_2^k(\beta_3)$$

and

$$H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_5) = H_1(s\backslash\alpha_2)-H_1(s\backslash\alpha_1) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3)$$

$$s_2^k(\beta_3) \leqslant \varepsilon_k s_1^k(\alpha_1)$$

$$s_2^k(\beta_3) \leqslant \varepsilon_k s_1^k(\alpha_5)$$

and in this way it is shown that the point (α_2,β_1) is a perfectly proper equilibrium point.

In an analogous way we will show that the point $(\mu\alpha+(1-\mu)\alpha_5,\beta_1)$ with $\mu \in [0,1]$ is a perfectly proper equilibrium point. Indeed take the sequences $\varepsilon_k = \frac{1}{k+2}$; $s_1^k(\alpha_1) = s_1^k(\alpha_5) =$

$= \frac{1}{2}\,[1 - \frac{300(k+2)+1}{[300(k+2)-1](k+2)}]$; $s_1^k(\alpha_2) = \frac{1}{k+2}$; $s_1^k(\alpha_3) = s_1^k(\alpha_4) =$

$= \frac{1}{300(k+2)^2}\,(1 + \frac{1}{300(k+2)-1})$; $s_2^k(\beta_1) = 1 - \frac{300(k+2)}{(300(k+2)-1)(k+2)^2}$;

$s_2^k(\beta_2) = \frac{1}{(k+2)^2}$; $s_2^k(\beta_3) = \frac{1}{(k+2)^2}\ \frac{1}{300(k+2)-1}$

then we have

$$H_1(s\backslash\alpha_5)-H_1(s\backslash\alpha_2) = H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_2) =$$

$$= \frac{1}{(k+2)^2} - \frac{1}{(k+2)^2}\ \frac{1}{300(k+2)-1}$$

$$H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_3) = H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_4) =$$

$$= 1 - \frac{1}{(k+2)^2} - \frac{1}{(k+2)^2} \quad \frac{1}{300(k+2)-1}$$

and

$$H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_2) < H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_3) = H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_4)$$

$$s_1^k(\alpha_3) \leqslant \varepsilon_k s_1^k(\alpha_2)$$

$$s_1^k(\alpha_4) \leqslant \varepsilon_k s_1^k(\alpha_2).$$

For the second player

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2) = 1 - \frac{1}{(k+2)^2}\left(1 + \frac{1}{300(k+2)-1}\right)$$

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3) = 1 - \frac{3}{300(k+2)^2}\left(1 + \frac{1}{300(k+2)-1}\right)$$

and

$$H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3)$$

$$s_2^k(\beta_3) \leqslant \varepsilon_k s_2^k(\beta_2).$$

Finally

$$H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_2) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2)$$

$$s_2^k(\beta_2) \leqslant \varepsilon_k s_1^k(\alpha_2)$$

$$H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_2) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3)$$

$$s_2^k(\beta_3) \leqslant \varepsilon_k s_1^k(\alpha_2)$$

and

$$H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_3) < H_2(s\backslash\beta_1)-H_2(s\backslash\beta_3)$$

$$s_2^k(\beta_3) \leqslant \varepsilon_k s_1^k(\alpha_3).$$

Similarly for $s_2^k(\beta_3) \leqslant \varepsilon_k s_1^k(\alpha_4)$.

The term $H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_3)$ equals $H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2)$ and

$H_1(s\backslash\alpha_1)-H_1(s\backslash\alpha_4)$ equals $H_2(s\backslash\beta_1)-H_2(s\backslash\beta_2)$.

The strategy α_5 does not have to appear in the inequalities since $s_1^k(\alpha_5) = s_1^k(\alpha_1)$.

Thus, the point $(\mu\alpha_1+(1-\mu)\alpha_5,\beta_1)$ with $\mu \in [0,1]$ is a perfectly proper equilibrium point.

We have that perfectly proper equilibrium points are not always s.p.p. equilibrium points. For suppose otherwise, and let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence satisfying the conditions of the definition and $\{s_k\}_{k=1}^{\infty} = \{(s_1^k,s_2^k)\}_{k=1}^{\infty}$ a sequence of strongly ε_k - perfectly proper equilibria converging to (α_2,β_1). Since $\lim_{k\to\infty} s_2^k(\beta_1) = 1$ and $\lim_{k\to\infty} s_2^k(\beta_2) = \lim_{k\to\infty} s_2^k(\beta_3) = 0$, $\lim_{k\to\infty} s_1^k(\alpha_2) = 1$; $\lim_{k\to\infty} s_1^k(\alpha_i) = 0 \quad i \in \{1,3,4,5\}$ $\exists\, K \in N$ such that $\forall\ k \geq K$, $\alpha_3,\alpha_4 \notin B_1(s_k)$. Hence $s_1^k(\alpha_3) = s_1^k(\alpha_4)$ if $k \geq K$, for the s_k are strongly ε_k - perfectly proper and α_3, α_4 are payoff-equivalent for player 1. Accordingly, for $k \geq K$

$$H_2(s_k\backslash\beta_1)-H_2(s_k\backslash\beta_2) < H_2(s_k\backslash\beta_1)-H_2(s_k\backslash\beta_3)$$

which successively entails that $s_2^k(\beta_3) \leq \varepsilon_k s_2^k(\beta_2)$.

This implies

$$H_1(s_k\backslash\alpha_2) < H_1(s_k\backslash\alpha_1)$$

which implies

$$H_1(s_k\backslash\alpha_1)-H_1(s_k\backslash\alpha_5) = 0 < H_1(s_k\backslash\alpha_1)-H_1(s_k\backslash\alpha_2)$$

and

$$s_1^k(\alpha_2) \leq \varepsilon_k s_1^k(\alpha_5).$$

But this means that it is impossible that $\lim_{k\to\infty} s_1^k(\alpha_2) = 1$ which is a contradiction. Hence s.p.p. equilibrium points is a strict refinement of the concept of perfectly proper equilibrium.

4. EXISTENCE OF S.P.P. EQUILIBRIUM POINT

In this paragraph we will prove that any normal n-person game has a s.p.p. equilibrium point. The proof follows the ideas of García Jurado.

We first show that $\forall\ \varepsilon_k \in (0,1)\ \ \exists$ a ε_k - spp equilibrium s_k.

Consider $\varepsilon_k \in (0,1)$. Denoting $|\Phi_i| = m_i$, define,

$$\forall\ i \in \{1,2,\dots,n\}, \quad \text{let } \gamma = \frac{\varepsilon_k^{\sum_{i=1}^{n} m_i}}{\sum_{i=1}^{n} m_i} .$$

Consider the set $S_i(\gamma) = \{s_i \in S_i / s_i(\phi_i) \geqslant \gamma_i \quad \forall\ \phi_i \in \Phi_i\}$

$\forall\ i \in \{1,2,\dots,n\}$

and let $S(\gamma) = S_1(\gamma) \times \dots \times S_n(\gamma)$.

Define now the multivalued function $F\colon S(\gamma) \longrightarrow P(S(\gamma))$ given by

$$F(s) = \{\sigma \in S(\gamma) \ /\ H_i(s\backslash\phi_i) - H_i(s\backslash\phi_i') < H_j(s\backslash\phi_j) - H_j(s\backslash\phi_j')$$

with $\phi_i \in B_i(s)$, $\phi_j \in B_j(s) \Rightarrow \sigma_j(\phi_j') \leqslant \varepsilon_k \sigma_i(\phi_i')\quad \forall\ \phi_i' \in \Phi_i$

$\forall\ \phi_j' \in \Phi_j$, $\forall\ i,j\ \ \sigma_i(\phi_i') = \sigma_i(\phi_i)$ payoff-equivalent for i,

$\phi_i', \phi_i \in \Phi_i \backslash B_i(s)\}$.

$S(\gamma)$ is compact convex and non empty and $F(s)$ is compact and convex $\forall\ s \in S(\gamma)$. Moreover $\forall\ s \in S(\gamma)$, $F(s)$ is non-empty. Indeed, if we define

$$A(s\backslash\phi_i') = \sum_{j=1}^{n} |\{\phi_j' \in \Phi_j / H_j(s\backslash\phi_j) - H_j(s\backslash\phi_j') < H_i(s\backslash\phi_i) - H_i(s\backslash\phi_i')$$

with $\phi_j \in B_j(s)$, $\phi_i \in B_i(s)\}|\quad \forall\ \phi_i' \in \Phi_i \quad \forall\ i \in \{1,\dots,n\}$

and $\sigma = (\sigma_1,\dots,\sigma_n)$ such that for each player i

$$\sigma_i(\phi_i) = \frac{\varepsilon_k^{A(s\backslash\phi_i)}}{\sum_{i=1}^{n} \sum_{\phi_i' \varepsilon \Phi_i} \varepsilon_k^{A(s\backslash\phi_i')}} \qquad \forall \ \phi_i \notin B_i(s)$$

$$\sigma_i(\phi_i) = \frac{1 - \sum_{\phi_i' \notin B_i(s)} \sigma_i(\phi_i')}{|B_i(s)|} \qquad \forall \ \phi_i \in B_i(s)$$

then $\sigma \in F(s)$ and this for each $s \in S(\gamma)$.

We observe that F is upper semicontinuous. By Kakutani's fixed point theorem (1941), F has a fixed point which is ε_k - s.p.p. Now let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence such that it is possible to find a subsequence of ε_k - s.p.p. equilibrium points.

Because S is compact this subsequence converges to a point $s \in S$ which is s.p.p. equilibrium point. (q.e.d.)

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