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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR DIFFUSION EQUATIONS OF POPULATION BIOLOGY WITH INITIAL DATA IN L^p SPACES

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ABSTRACT. The paper considers the existence and uniqueness of weak global and local solutions of some nonlinear diffusion equations that arise in the study of population biology or ecological phenomena in three dimensional Euclidean space. The results are obtained in a class of special functions defined on some L^p spaces when the initial data are also in these L^p spaces. The interest of the techniques employed here relies on the fact that it is by succesive approximations and hence amenable to numerical treatment.

1. INTRODUCTION

This paper is primarily an application of potential operator's theory and some results of maximal functions to the study of nonlinear diffusion equations evolving from biological and ecological phenomena in three-dimensional spaces as pioneered in [1]. In essense, it is an extension of the work in [1] to higher order nonlinearity.

Throughout the paper we consider a population evolving in a bounded three-dimensional habitat. In what follows $x = (x_1,x_2,x_3)$ will be a point in $\mathbf{R}^3$ and $u(x,t)$ will denote the population density at time t at point x. $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ will take the usual meaning of the distance of the point x from the origin. At this point we wish to study two types of nonlinear diffusion problems that arise in biology or ecology. The Von Bertalanffy type and Hoppensteadt's system of nonlinear diffusion equations arising in oscillatory media. That is, we will consider, respectively, the initial value nonlinear diffusion equation

$$u_t(x,t)-\nabla^2 u = a(x)^3 u-b(x)u^3 \qquad (1.1)$$

$$u(x,0) = f(x) \qquad (1.2)$$

where a(x) and b(x) are continuous functions of the spatial variable x and represent the intrinsic growth and death rate, respectively, due to interaction of population within the bounded habitat under consideration, and the system of initial value nonlinear diffusion equations

$$u_t(x,t)-\nabla^2 u = \varepsilon_0(x)^3 u - \frac{\varepsilon(x)u^3}{3} + C(x)^3 v \qquad (1.3)$$

$$v_t(x,t)-\nabla^2 v = -C_0(x)^3 u \qquad (1.4)$$

$$u(x,0) = F(x) \quad ; \quad v(x,0) = G(x) \qquad (1.5)$$

where $C(x)$, $C_0(x)$ are continuous functions of x and ε_0 depends on $\varepsilon(x)$ which is assumed to be continuous and small valued in x and has small norm in the spaces under consideration.

This paper is intended to answer the problem [1]: "Suppose the density distribution of a population u is known throughout $\mathbf{R}^3$ and, at an instant t=0, a new bounded habitat G, say, opens for the species to migrate into it. If we assume a generalized logistic growth for the species u as well as a predatorial action (death or decay) within G, find the density distribution of the population u in G for all time $t > 0$, assuming that migration is governed by diffusion". As a simplification, we neglect the description of any natural barrier beyond G, assuming that any diffusion of biomass toward infinity can be interpreted as a loss due to inhospitable sub-habitats and, as such, it is natural for us to consider the problem on the whole of $\mathbf{R}^3$.

If we now let $W(x,t) = (4\pi t)^{-3/2}\exp\{-|x|^2/4t\}$ be the fundamental solution to equation (1.1) and $\otimes$ be the convolution symbol in time and space and $*$ the convolution in space, then a formal solution to equation (1.1) subject to the initial condition (1.2) is

$$u(x,t) = W \otimes [a^3u-bu^3] + W * f. \tag{1.6}$$

We define the mapping ϕ on $\mathbf{R}^3 \times [0,T)$ by

$$(\phi u)(x,t) = W \otimes [a^3u-bu^3] + W * f. \tag{1.7}$$

Then by using the method of succesive approximations we can show that the iterations

$$u_{\kappa+1} = \phi(u_\kappa)$$

converge in the norm of definition to the solution of the integral equation (1.6) and that the mapping ϕ is a contraction from a ball of radius, say, r_0, into itself. Similarly if W represents a fundamental solution to (1.3), (1.4) (which may differ only by a constant), then formal solutions to equations (1.3), (1.4) subject to the initial conditions (1.5) are

$$u(x,t) = W \otimes [\varepsilon_0^3u-\frac{\varepsilon}{3}u^3 + C^3v] + W * F \tag{1.8}$$

$$v(x,t) = -W \otimes C_0^3u + W * G. \tag{1.9}$$

We then define the mapping Ψ by

$$\Psi(u,v)(x,t) = W \otimes [\varepsilon_0^3u-\frac{\varepsilon}{3}u^3 + C^3v] + W * F \tag{1.10}$$

where $v = -W \otimes C_0^3u + W * G$.

Then by using the method of successive approximations we can show that the iterations

$$u_{\kappa+1} = \Psi(u_\kappa)$$

converge in the norm of definition to the solution of the integral equations (1.8), (1.9) and that the mapping is a contraction from a ball of radius, say, r_t, into itself. Here

$\Psi(u_K) = \Psi(u_K, u_K)$.

In addition to addresing the posed problem, we will provide existence and uniqueness of both weak global solutions (global in time) and local solutions (local in time) using the techniques of [1,4,5].

2. CLASS OF FUNCTIONS AND FIXED POINT PROPERTY

We will consider measurable functions defined on $\mathbf{R}^3 \times [0,T)$ for which

$$\|u^*\|_p = \left(\int_{\mathbf{R}^3} (\sup_{0<t<T} |u(x,t)|)^p dx\right)^{1/p} < \infty$$

where $u^*(x) = \sup_{0<t<T} |u(x,t)|$.

For the initial data, we use the usual L^p norm

$$\|f\|_p = \left(\int_{\mathbf{R}^3} |f|^p dx\right)^{1/p}.$$

As in [1,2,3], we also consider the following standard estimates for the fundamental solution W as

$$|W| \leqslant \beta [|x| + t^{1/2}]^{-3} \quad ; \quad \beta > 0 \text{ is a constant} \qquad (2.1)$$

and

$$|W| \leqslant \frac{C}{|x|^{3-\theta}\, t^{\theta/2}} \quad ; \quad 0 < \theta < 2 \text{ and } C > 0 \text{ is a constant} \qquad (2.2)$$

In this class of functions, a solution u of (1.6) for all $t > 0$ will be called a weak global solution of (1.1), (1.2) whenever the integrals that are involved exist in the Lebesgue sense for all values of $t > 0$. This definition similarly applies to the system of diffusion equations (1.3),(1.4),(1.5) when their formal solutions are given as in (1.8),(1.9).

FIXED POINT PROPERTIES

In what follows we are going to consider Banach spaces of Lebesgue measurable functions defined on $\mathbf{R}^3 \times \mathbf{R}_+$ for which the perturbed operator ϕ is defined by

$$\phi(u_1,u_2,u_3) = W \otimes [a^3 u_1 - b.u_1.u_2.u_3] + W * f \qquad (2.3)$$

(where $u_i(x,t)$, $i = 1,2,3$ are Lebesgue measurable functions on $\mathbf{R}^3 \times \mathbf{R}_+$) and satisfies the estimate

$$\|\phi(u_1,u_2,u_3)\| \leqslant c_1\|u_1\|.\|u_2\|.\|u_3\| + c_2\|u_1\| + \gamma. \qquad (2.4)$$

The norm in (2.4) is that of the Banach space in question.

LEMMA 1. *Let* $\phi(u_1,u_2,u_3)$ *be a general operator of the type* (2.3), *mapping the product space* B×B×B *into* B, *where* B *denotes a Banach space satisfying the estimate* (2.4). *Suppose that* c_1, c_2 *and* γ *satisfy* $c_1 > 0$, $0 \leqslant c_2 < 1$, $0 \leqslant \gamma$ *is small. Then for* $u_1 = u_2 = u_3 = u$, *there exists* $\delta > 0$ *such that if* $\gamma < \delta$ *the mapping* $\phi(u) = \phi(u,u,u)$ *possesses one and only one fixed point* u_0 *in a ball of radius* y_1. *Furthermore,*

(a) $\|u_0\| \leqslant y_1$ *where* y_1 *is the smallest positive root of*

$$y = c_1 y^3 + c_2 y + \gamma.$$

(b) $\|u_0\| \to 0$ *as* $\gamma \to 0$.

We will prove this Lemma through the following Lemma:

LEMMA 2. *Given* $c_1 > 0$, $0 \leqslant c_2 < 1$ *and a small* $\gamma \geqslant 0$, *the cubic polynomial* $f(y) = c_1 y^3 + c_2 y + \gamma$ *has at least one positive fixed point. Furthermore, if we call* y_1 *the smallest positive fixed point of* f *depending on* γ *then* $y_1(\gamma) \to 0$ *as* $\gamma \to 0$.

Proof. For $\gamma = 0$, $y = c_1 y^3 + c_2 y$. Then clearly $y_1 = 0$ and $y_2 = (\frac{1-c_2}{c_1})^{1/2}$ are the two non-negative fixed points and we are done.

Now due to the fact that $0 \leqslant c_2 < 1$, the graph of $z = c_1 y^3 + c_2 y$ is underneath the line $z = y$ for $0 < \tilde{y}_1 < y < \tilde{y}_2$. This implies that there exists a $\delta > 0$ small enough such that if $0 < \gamma < \delta$

a portion of the graph $f(y) = c_1y^3 + c_2y + \gamma$ is still underneath the line $z = y$. A portion such that for some pair y_1, y_2 we have $0 < y_1 < y < y_2$. We may select y_1 and y_2, the optimal ones, that is, the two positive roots of $y = c_1y^3 + c_2y + \gamma$. This automatically implies that for $0 < \gamma < \delta$, $y_1 = y_1(\delta)$, thus $y_1(\gamma) \to 0$ as $\gamma \downarrow 0$. This concludes the proof.

Proof of Lemma 1. For $0 \leqslant c_2 < 1$, $c_1 > 0$ and a small $\gamma \geqslant 0$, we have from Lemma 2, that $f(y) = c_1y^3 + c_2y + \gamma = y$ has a smallest positive root $y_1(\gamma)$ such that $\gamma < \delta$, $\delta > 0$ is very small. For $y \in (0,y_1)$, we have $y \leqslant y_1$ and so $c_1y^3 + c_2y + \gamma \leqslant y_1$. Set $y = \|u\|$, then $c_1\|u\|^3 + c_2\|u\| + \gamma \leqslant y_1$ whenever $\|u\| \leqslant y_1$. Thus ϕ maps a ball of radius y_1 into itself. Furthermore, for $y \in (0,y_1)$ $f'(y) = 3c_1y^2 + c_2$ and $f''(y) = 6c_1y$ are positive. Thus $f(y)$ is monotonic increasing and convex in $(0,y_1)$. Hence we can find $\alpha_1, \alpha_2 \in (0,y_1)$ such that $\|f(\alpha_1)-f(\alpha_2)\| \leqslant f'(y)\|\alpha_1-\alpha_2\|$. Now set $\alpha_1 = u$, $\alpha_2 = w$ in a ball of radius y_1 and $\phi = f$, then we have $\|\phi(u)-\phi(w)\| \leqslant [3c_1y_1^2 + c_2]\|u-w\| = \lambda\|u-w\|$.

Since y_1 is very small, so is y_1^2 and so $3c_1y_1^2 + c_2 = \lambda < 1$. Hence ϕ is a contraction mapping in a ball of radius y_1 and, by the contraction mapping theorem, ϕ has a unique fixed point u_0 belonging to this ball. Hence $\|u_0\| \leqslant y_1$. Finally, from Lemma 2, $y_1 \to 0$ as $\gamma \downarrow 0$ implies that $\|u_0\| \to 0$ as $\gamma \downarrow 0$. This concludes the proof of Lemma 1.

3. ESTIMATE FOR THE INTEGRAL OPERATORS

In order for us to establish the desired results we will provide estimates of the integral operators of (1.7) and (1.10) in the form of Lemmas.

LEMMA 3. *Let* u(x,t) *satisfy* (1.6). *Then if* $f,a,b \in L^{9/2}(\mathbf{R}^3)$ we *have* $\|\phi(u)^*\|_{9/2} \leqslant c_1\|b\|_{9/2}\|u^*\|_{9/2}^3 + c_2\|a\|_{9/2}^3\|u^*\|_{9/2} + c_3\|f\|_{9/2}$ *where* c_1, c_2, c_3 *are positive constants.*

Proof. From (1.7) we have

$$|(\phi u)(x,t)| \leqslant \int_{\mathbf{R}^3}\int_0^t |W(x-y,t-\tau)|\,|a(y)|^3|u(y,\tau)|\,d\tau dy +$$

$$+ \int_{\mathbf{R}^3}\int_0^t |W(x-y,t-\tau)|\,|b(y,\tau)|\,|u(y,\tau)|^3 d\tau dy + |W * f|.$$

Then on taking the supremum over $t > 0$ and invoking estimate (2.1) we have

$$(\phi u)^*(x) \leqslant \beta_0\int_{\mathbf{R}^3} \frac{|a(y)|^3 u^*(y)}{|x-y|^{3-2}}\,dy + \beta_0\int_{\mathbf{R}^3} \frac{|b(y)|\,u^*(y)^3}{|x-y|^{3-2}}\,dy +$$

$+ \sup_{t>0} |W * f|$ where β_0 does not exceed $\beta\int_0^\infty (1+t^{1/2})^{-3}dt.$

If we now let $|a(y)|^3 u^*(y) \in L^{9/4}(\mathbf{R}^3)$ and $|b(y)|\,u^*(y)^3 \in$ $\in L^{9/4}(\mathbf{R}^3)$, then for p,q such that $1/q = 4/p - 2/3$, we apply the Hardy-Littlewood-Sobolev theorem to the first and second terms on the right - hand side to get

$$\|(\phi u)^*\|_q \leqslant \beta_0 A(q,\tfrac{p}{4})\|\,|a(.)|^3u^*(.)\|_{p/4} + \beta_0\tilde{A}(q,\tfrac{p}{4})\|\,|b(.)|u^*(.)^3\|_{p/4} +$$

$+ \|f^*\|_q$ where $f^*(x) = \sup_{t>0} |W * f|$ and $A(q,\frac{p}{4})$, $\tilde{A}(q,\frac{p}{4})$ are the constants resulting from the Hardy-Littlewood-Sobolev theorem. Finally, applying Schwarz's inequality twice to each of the first and second terms on the right - hand side and on noting that $\|f^*\|_q \leqslant B_q\|f\|_q$ we have

$$\|(\phi u)^*\|_q \leqslant C_1\|b\|_p\|u^*\|_p^3 + C_2\|a\|_p^3\|u^*\|_p + B_q\|f\|_q \qquad (3.1)$$

where $C_1 = \beta_0\tilde{A}(q,\frac{p}{4})$, $C_2 = \beta_0 A(q,\frac{p}{4})$.

Now if $p = q$ then from $1/q = 4/p - 2/3$ we see that $p = q = 9/2$. On substituting $p = q = 9/2$ into (3.1) and noting that $C_3 = B_{9/2}$, we have the required result.

LEMMA 4. *Let* u,v *satisfy* (1.8),(1.9). *Then if* $\varepsilon_0, \varepsilon, C(x), C_0(x)$, F,G *all belong to* $L^{9/2}(\mathbf{R}^3)$ *we have*

$$\|\Psi(u,v)^*\|_{9/2} \leq d_1\|\varepsilon\|_{9/2}\|u^*\|_{9/2}^3 + d_2\|\varepsilon_0\|_{9/2}^3\|u^*\|_{9/2} + d_3\|C\|_{9/2}^3\|v^*\|_{9/2} +$$

$$+ d_4\|F\|_{9/2} \quad \textit{and} \quad \|v^*\|_{9/2} \leq d_5\|C_0\|_{9/2}^3\|u^*\|_{9/2} + d_6\|G\|_{9/2}.$$

Proof. As in the proof of Lemma 3, we have from (1.10) that

$$\Psi(u,v)^* \leq \beta_0\int_{\mathbf{R}^3}|x-y|^{2-3}|\varepsilon_0(y)|^3u^*(y)dy + \frac{\beta_0}{3}\int_{\mathbf{R}^3}|x-y|^{2-3}|\varepsilon(y)|u^*(y)^3dy +$$

$$+ \beta_0\int_{\mathbf{R}^3}|x-y|^{2-3}|C(y)|^3v^*(y)dy + \sup_{t>0}|W * F|.$$

Now if we let $|\varepsilon_0(y)|^3u^*(y), |\varepsilon(y)|u^*(y)^3, |C(y)|^3v^*(y) \in L^{p/4}(\mathbf{R}^3)$ then, for p, q such that $1/q = 4/p - 2/3$, we have on applying the Hardy-Littlewood-Sobolev theorem on the first three terms of the right - hand side after taking the L^q norm of both sides

$$\|\Psi(u,v)^*\|_q \leq \beta_0 D(q,\tfrac{p}{4})\|\,|\varepsilon_0(.)|^3u^*(.)\|_{p/4} + \frac{\beta_0}{3}\tilde{D}(q,\tfrac{p}{4})\|\,|\varepsilon(.)|u^*(.)^3\|_{p/4} +$$

$$+ \beta_0\acute{D}(q,\tfrac{p}{4})\|\,|C(.)|^3v^*(.)\|_{p/4} + E_q\|F\|_q.$$

Applying the Schwarz's inequality twice on each of the first three terms on the right - hand side we have

$$\|\Psi(u,v)^*\|_q \leq d_1\|\varepsilon\|_p\|u^*\|_p^3 + d_2\|\varepsilon_0\|_p^3\|u^*\|_p + d_3\|C\|_p^3\|v^*\|_p + d_4\|F\|_q \quad \text{where}$$

$$d_1 = \frac{\beta_0}{3}\tilde{D}(\tfrac{p}{4},q)\ , \ d_2 = \beta_0 D(\tfrac{p}{4},q)\ , \ d_3 = \beta_0\acute{D}(\tfrac{p}{4},q)\ , \text{ and } d_4 = E_q.$$

On letting $q = p$ in $1/q = 4/p - 2/3$, we have $p = q = 9/2$ and the first estimate follows. By invoking the same approach on $v(x,t) = -W \otimes C_0^3u + W * G$, we have

$\|v^*\|_{9/2} \leqslant d_5\|C_0\|^3_{9/2}\|u^*\|_{9/2} + d_6\|G\|_{9/2}$ where again $d_5 = \beta_0\tilde{B}(\frac{p}{4},q)$, and $d_6 = \tilde{E}_q$, are the Hardy-Littlewood-Sobolev constants. This completes the proof of Lemma 4.

STATEMENT OF RESULTS

The main results of this section are contained in the following theorems.

THEOREM 1. *Suppose that* $f,a,b \in L^{9/2}(\mathbf{R}^3)$, *and furthermore, suppose that* $\beta\ A(\frac{9}{2},\frac{9}{8}) < \|a\|^{-3}_{9/2}$. *Then if* $B(9/2)\|f\|_{9/2} \leqslant \delta$; $\delta > 0$ *is very small. Equation* (1.1) *with initial data* (1.2) *possesses one and only one solution* $u(x,t)$ *such that* $\|u^*\|_{9/2} \leqslant y_0$, *where* y_0 *depends on* δ.

THEOREM 2. *Suppose that* $\varepsilon_0, \varepsilon, C(x), C_0(x), F, G$ *all belong to* $L^{9/2}(\mathbf{R}^3)$, *and furthermore that* $\|F\|_{9/2}$, $\|G\|_{9/2}$ *are small such that there exists a small* $\delta_0 > 0$ *satisfying*

$$(d_3d_6\|C\|^3_{9/2}\|G\|_{9/2} + d_4\|F\|_{9/2}) \leqslant \delta_0.$$

Then if $(d_2\|\varepsilon_0\|^3_{9/2} + d_3d_5\|C\|^3_{9/2}\|C_0\|^3_{9/2}) < 1$, *the system of initial value problem* (1.3)-(1.5) *has one and only one pair of weak global solutions* $u(x,t)$, $v(x,t)$ *satisfying*

(a) $\|u^*\|_{9/2} \leqslant s_0$ *and* (b) $\|v^*\|_{9/2} \leqslant b_0$

where s_0 *and* b_0 *depend on* δ_0, $(d_2\|\varepsilon_0\|^3_{9/2} + d_3d_5\|C\|^3_{9/2}\|C_0\|^3_{9/2})$. and $d_1\|\varepsilon\|_{9/2}$.

Proof of Theorem 1. If suffices to prove that the perturbed integral operator (1.7) possesses a unique solution in the $L^{9/2}(\mathbf{R}^3)$ norm. By Lemma 3, we see that ϕ satisfies the inequality (2.4).

Since $B(9/2)\|f\|_{9/2} \leq \delta$, $\delta > 0$ is very small and $\beta_0 A(\frac{9}{2},\frac{9}{8}) < \|a\|^3_{9/2}$ implies that $\beta_0 A(\frac{9}{2},\frac{9}{8})\|a\|^3_{9/2} < 1$. On letting $\gamma = B(9/2)\|f\|_{9/2}$ and $C_2 = \beta_0 A(\frac{9}{2},\frac{9}{8})\|a\|^3_{9/2}$ we see from Lemma 1 that ϕ possesses one and only one solution $u(x,t)$ satisfying $\|u^*\|_{9/2} \leq y_0$, where y_0 depends on δ.

Proof of Theorem 2. It suffices to prove that the perturbed operator Ψ of (1.10) possesses a unique pair of solutions defined in the $L^{9/2}(\mathbf{R}^3)$ norm. Injecting the second inequality of Lemma 4 into the first we have

$$\|\Psi(u,u)^*\|_{9/2} \leq d_1\|\varepsilon\|_{9/2}\|u^*\|^3_{9/2} + [d_2\|\varepsilon_0\|^3_{9/2} + d_3d_5\|C\|^3_{9/2}\|C_0\|^3_{9/2}]\|u^*\|_{9/2} + (d_3d_6\|C\|^3_{9/2}\|G\|_{9/2} + d_4\|F\|_{9/2}).$$

Let $\alpha_1 = d_1\|\varepsilon\|_{9/2}$, $\alpha_2 = [d_2\|\varepsilon_0\|^3_{9/2} + d_3d_5\|C\|^3_{9/2}\|C_0\|^3_{9/2}]$ and $\gamma = [d_3d_6\|C\|^3_{9/2}\|G\|_{9/2} + d_4\|F\|_{9/2}]$. Then we see that $\alpha_1 > 0$ and by the hypothesis of the theorem $0 < \alpha_2 < 1$ and $\gamma \leq \delta_0$. Hence by Lemma 1 $\Psi(u) = \Psi(u,u)$ possesses one and only one solution u in a ball of radius $s_0 = s_0(\delta_0)$ such that $\|u^*\|_{9/2} \leq \delta_0$. Then from $\|v^*\|_{9/2} \leq d_5\|C_0\|^3_{9/2}\|u^*\|_{9/2} + d_6\|G\|_{9/2}$ we have $\|v^*\|_{9/2} \leq d_5\|C_0\|^3_{9/2}s_0 + d_6\|G\|_{9/2}$. Since $\|G\|_{9/2}$ is assumed small, $d_5\|C_0\|^3_{9/2}s_0$ can be made as small as possible so that there exists a small b_0 such that $d_5\|C_0\|^3_{9/2}s_0 + d_6\|G\|_{9/2} \leq b_0$. Hence $\|v^*\|_{9/2} \leq b_0$, the uniqueness of v depending on u. This concludes the proof of the theorem.

4. EXISTENCE AND UNIQUENESS OF LOCAL SOLUTIONS

In this section we will consider the case when the coefficients of problems (1.1)-(1.2) and (1.3)-(1.5) belong to $L^\infty(\mathbf{R}^3)$. We shall see that in order to achieve the conditions of Lemma 1 we need only consider the solutions for small time T (the size of T determined by Lemma 1).

Now if $a(x)^3 = a_1(x)$, $b(x) \in L^\infty \cap L^p(\mathbf{R}^3)$; $3 < p < \infty$ and $f \in L^{6/\theta}(\mathbf{R}^3)$, we have from (1.7) that

$$|(\phi u)(x,t)| \leqslant \|a_1\|_\infty |W \otimes u| + \|b\|_\infty |W \otimes u^3| + |W * f|.$$

If we let M(u) be the maximal function then

$$\sup_{0<t<T} |(\phi u)(x,t)| \leqslant \|a_1\|_\infty T M(u^*) + C_\vartheta T^{1-\frac{\theta}{2}} \|b\|_\infty \int_{\mathbf{R}^3} \frac{u^*(y)}{|x-y|^{3-\theta}}\, dy \; +$$

$$+ \sup_{0<t<T} |W * f|$$

where we have utilized estimate (2.2). If we now let $u^*(y)^3 \in L^{p/3}(\mathbf{R}^3)$, for p such that $1/p = 3/p - \theta/3$; $0 < \theta < 2$, we have on taking the L^p norm of both sides after invoking the Hardy-Littlewood-Sobolev theorem that

$$\|(\phi u)^*\|_p \leqslant \|a_1\|_\infty T \|M(u^*)\|_p + C_\vartheta T^{1-\frac{\theta}{2}} \|b\|_\infty \|u^*(.)^3\|_{p/3} + \|\sup_{0<t<T} |W * f|\|_p .$$

Since $f \in L^p(\mathbf{R}^3)$, we have that

$$\|\phi(u)^*\|_p \leqslant \|a_1\|_\infty T \|u^*\|_p + C_\theta T^{1-\frac{\theta}{2}} \|b\|_\infty \|u^*\|_p^3 + A_p \|f\|_p \; ; \; p = 6/\theta; \; 0 < \theta < 2. \quad (4.1)$$

Then the following theorem holds:

THEOREM 3. *Suppose that* $a(x)^3, b(x) \in L^\infty \cap L^p(\mathbf{R}^3)$, $3 < p < \infty$ *and* W *satisfies estimate* (2.2). *Suppose further that* $f \in L^{6/\theta}(\mathbf{R}^3)$ *and that the norm* $\|f\|_{6/\theta}$ *is very small, then equations* (1.1)-(1.2) *admit a unique solution* $u(x,t)$ *defined on* $\mathbf{R}^3 \times (0,T)$ *for very small values of* T, *satisfying* $\|u^*\|_{6/\theta} \leqslant s$, *where* s *depends on* T *and* $\|f\|_{6/\theta}$.

Proof. From inequality (4.1) we see that if we let $\alpha_1 = \|a_1\|_\infty T$, $\alpha_2 = C_\theta T^{1-\frac{\theta}{2}}\|b\|_\infty$ and $\gamma = A_{6/\theta}\|f\|_{6/\theta}$, $0 < \theta < 2$, then for all values of T satisfying $T^{1-\frac{\theta}{2}} < \frac{1}{C_\theta\|b\|_\infty}$ we can see that $0 < \alpha_2 < 1$ and $\alpha_1 > 0$, so that if $A_{6/\theta}\|f\|_{6/\theta}$ is small, we conclude by Lemma 1, that the integral equation (1.7) possesses one and only one solution u in a ball of radius s such that $\|u^*\|_{6/\theta} \leqslant s$ where s depends on T and $\|f\|_{6/\theta}$. This completes the proof of the theorem.

Consider the system of equations (1.3)-(1.5). If we also assume that $\varepsilon_0(x)^3 = \varepsilon(x), C(x)^3, C_0(x)^3$ all belong to $L^\infty \cap L^p(\mathbf{R}^3)$, $3 < p < \infty$, then we have as estimate of the integral equation (1.10)

$$\|\Psi(u,v)^*\|_{6/\theta} \leqslant \|\varepsilon\|_\infty T\|u^*\|_{6/\theta} + T\|\tilde{C}\|_\infty\|v^*\|_{6/\theta} + d_\theta T^{1-\frac{\theta}{2}}\|\varepsilon\|_\infty\|u^*\|^3_{6/\theta} + \\ + B_{6/\theta}\|F\|_{6/\theta} \;;\quad 0 < \theta < 2 \tag{4.2}$$

and

$$\|v^*\|_{6/\theta} \leqslant T\|\tilde{C}_0\|_\infty\|u^*\|_{6/\theta} + E_\theta\|G\|_{6/\theta} \tag{4.3}$$

where we have replaced $C(x)^3$ by $\tilde{C}(x)$ and $C_0(x)^3$ by $\tilde{C}_0(x)$. Inequalities (4.2) and (4.3) are derived in the same manner as inequality (4.1).

THEOREM 4. *Let* $\varepsilon_0(x) = \varepsilon(x), C(x)^3, C_0(x)^3$ *all belonging to* $L^\infty \cap L^p(\mathbf{R}^3)$, $3 < p < \infty$, *and* W *satisfying the estimate* (2.2). *We suppose further that* $F, G \in L^{6/\theta}(\mathbf{R}^3)$ *such that both* $\|F\|_{6/\theta}$ *and* $\|G\|_{6/\theta}$ *are very small. Then the system of nonlinear diffusion equations* (1.3)-(1.5) *admits a pair of unique solutions* $U(x,t), v(x,t) \in \mathbf{R}^3 \times (0,T)$ *for very small values of* T *and satisfies* $\|u^*\|_{6/\theta} \leqslant r_0$, $\|v^*\|_{6/\theta} \leqslant r_1$, *where* r_0 *and* r_1 *depends on* T, $\|F\|_{6/\theta}$ *and* $\|G\|_{6/\theta}$.

The proof of this theorem follows from inequalities (4.2) and (4.3) and is concluded in the same manner as theorem 2.

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