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SPECTRA OF ANALYTIC FUNCTIONS IN INFINITE DIMENSIONS

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This article is based on a talk which describes part of work by R.M. Aron, B. Cole, and T. Gamelin, entitled "Spectra of analytic functions on a Banach space," which appeared in [A-C-G]. We begin by recalling the one dimensional situation. Let D be the complex open unit disc. $H^\infty(D)$ denotes the algebra of bounded analytic functions on D and $A(D)$ denotes the subalgebra of analytic functions on D which are continuous on $\bar{D}$. Both are Banach algebras when given the norm $\|f\| = \sup_{z\in D}|f(z)|$. If we call either of these algebras A, then we can define $M(A)$ to be the "maximal ideal space" of A, that is the space of (automatically continuous) homomorphisms $\phi : A \to C$. We regard $M(A)$ as a subset of the dual of A, A'. It is not hard to prove that $M(A)$ is a weak-star closed subset of the unit ball of A', and thus $M(A)$ is weak-star compact. For each $a \in D$, $\delta(a) = \delta_a \in M(A)$, where $\delta_a(f) \equiv f(a)$. Thus, we see that $D \subset M(A)$. Next, let $\pi : M(A) \to C$ be the mapping which takes a homomorphism ϕ to $\pi(\phi) = \phi(z)$. One shows that

i. π is continuous,

ii. $\pi \circ \delta(a) = a$ for all $a \in D$, and that

iii. $|\pi(\phi)| \leq 1$ for all $\phi \in M(A)$.

It follows easily that π is onto the closed disc $\bar{D}$. For each $a \in \bar{D}$, we let M_a denote $\{\phi \in M(A) \mid \pi(\phi) = a\}$. As specific examples, one can show that if $A = A(D)$ then each $M_a = \{\delta_a\}$, and that $M(A(D))$ is homeomorphic via δ to $\bar{D}$. On the other hand, the situation for $A = H^\infty(D)$ is much more complicated. Although $M_a = \{\delta_a\}$ for $a \in D$ so that π is one-to-one over D, one needs the axiom of choice to "exhibit" a homomorphism $\phi \in M_1$ (which we know to be non-empty by the above remarks). Thus, even in this situation, the structure of $M(H^\infty(D))$ is far from clear.

Our attention here will be focused on the spectra of analogous algebras of analytic functions defined on the open unit ball B_E of a complex Banach space E. In this talk, we'll confine ourselves to describing the basic spaces of interest and giving examples which, hopefully, explain the problems involved and the reasons for our interest. So, to begin, let's define what we mean by an *analytic function* $f : B \to C$, where B is an open ball, center $O \in E$. f is said to be *analytic* on B if f has a complex Fréchet derivative at each point of E or, equivalently, there is a sequence $(P_n)_{n=0}^{\infty}$ of complex-valued continuous n-homogeneous polynomials such that $f(x) = \sum_{n=0}^{\infty} P_n(x)$ for all $x \in B$. For example, if $E = c_0$ and $B = E$, let $f(x) = \sum_{n=0}^{\infty} (x_n)^n$. A routine argument shows that f is analytic but that $\|f\|_{B_R} = \sup \{|f(x)| \mid \|x\| < R\} = \infty$ for every $R \geq 1$. Let $H^\infty(B_E) = \{f : B_E \to C \mid f$ is analytic and bounded on $B_E\}$, and let $A(B_E) = \{f : B_E \to C \mid f$ is analytic and uniformly continuous on $B_E\}$. Note that $A(B_E)$ is a subalgebra of $H^\infty(B_E)$, and that both are Banach algebras with norm $\|f\| = \sup\{|f(x)| \mid x \in B_E\}$. The above example shows that even if the function f is analytic on the entire space, it is possible that $\|f\| = \infty$. Thus, we must impose an *additional* (albeit natural) condition on the subalgebra $A(B_E)$, such as *uniform continuity*, in order to ensure that $\|f\|$ is always finite.

We let A denote either of the algebras $H^\infty(B_E)$ or $A(B_E)$, and we define $M(A)$ to be the space of homomorphisms $\phi : A \to C$. $M(A)$ is a weak-star compact subset of A' and, as in the one dimensional case, there is a natural embedding δ of B_E into $M(A)$ which takes a point $a \in B_E$ to δ_a. Next, let $\pi : M(A) \to E''$ be defined by $\pi(\phi) = \phi|_{E'}$ (note that E' is a subspace of A for either of our algebras A). Since the analogues of (i), (ii), (iii) above hold, we conclude that π is a continuous surjection of $M(A)$ onto $\overline{B_E}$. We let M_a denote the fiber over a, that is the set of homomorphisms $\phi \in M(A)$ such that $\pi(\phi) = a$.

Example 1. $A = A(B_{c_0})$. This algebra has the property that every $f \in A$ is uniformly weakly continuous on B_{c_0}, since one can show that the subalgebra generated by $\ell_1 = c_0'$ is dense in A. As a result, every such f can be extended to a weak-star uniformly continuous analytic function on the weak-star compact set $\overline{B_{\ell_\infty}}$. From this it follows that $M(A) \cong (\overline{B_{\ell_\infty}}, \sigma(\ell_\infty, \ell_1))$.

Example 1 is somewhat deceptive, as Example 2 shows. Here, we examine $M(A(B_{\ell_2}))$ and we show that this set is much, much larger that $\overline{B_{\ell_2}}$.

Example 2. Let $E = \ell_2$ and let $I = \{f \in A(B_{\ell_2}) \mid f(e_n) = 0$ for all $n \geq$ some $n(f) \in N\}$. It is easy to see that I is a proper ideal in $A(B_{\ell_2})$, and so I is contained in a maximal ideal, which by the Gelfand-Mazur theorem is the kernel of some $\phi \in M(A(B_{\ell_2}))$. Note that $\pi(\phi) = (\phi(z_1), \phi(z_2), \ldots)$ where $z_j \in A(B_{\ell_2})$ is defined by $z_j(x) = x_j$. Since each z_j

belongs to I, we conclude that $\pi(\phi) = 0$. On the other hand, $\phi \neq \delta_a$ for any $a \in \ell_2$. In fact, if $\phi = \delta_a$, say, fix $k \in N$ and define $f_k(x) = 1 - \sum_{n=k}^{\infty} x_n^2$. Since $f_k \in I$, it follows that $\phi(f_k) = \delta_a(f_k) = 1 - \sum_{n=k}^{\infty} a_n^2 = 0$, which leads to a contradiction. Thus, even in the fiber over 0, we have many homomorphisms. Moreover, the Shilov boundary of such an algebra A can in fact intersect the fiber over 0 (see [A-C-G]).

The situation concerning M_0 is considerably more complicated, as the next example shows.

Example 2′. Let A be as in Example 2. An application of the finite intersection property to the compact set $M(A)$ shows that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \{\delta_{e_j}\}}$ is non-empty. Let ϕ belong to this intersection. Using only the definition of weak-star closure, it is straightforward that $\pi(\phi) = 0$. Also, given any sequence $\alpha = (\alpha_n) \in \ell_\infty$ the polynomial $P_\alpha : \ell_2 \to C$ given by $P_\alpha(x) = \sum_{j=1}^{\infty} \alpha_j x_j^2$ is in A. Since $\hat{P}_\alpha(\delta_{e_j}) = \delta_{e_j}(P_\alpha) = \alpha_j$ for all j, we see that the set $\{\delta_{e_j}\}$ "behaves" like $N \subset \beta N$. In particular $\overline{\{\delta_{e_j}\}}^{M(A)} \cong \beta N$, so that M_0 contains a copy of $\beta N \backslash N$.

In fact, a somewhat more elaborate argument shows that when $A = H^\infty(B_E)$ for an arbitrary infinite dimensional Banach space E, for every $x \in \overline{B_E}$ M_x contains a copy of $\beta N \backslash N$.

Example 3. As our last example, we outline an explicit manner to obtain homomorphisms ϕ lying in $M_{x''}$ where $x'' \in B_{E''}$. To fix our ideas, let $A = H^\infty(B_E)$ and let $f = \sum P_n \in A$. Each P_n corresponds to a unique symmetric n-linear continuous mapping $A_n : E \times \ldots \times E \to C$. Each A_n has a canonical extension to a continuous n-linear mapping $A_n : E'' \times \ldots \times E'' \to C$. (See [A-B] and [D-G]). To illustrate, let's take the case $n = 2$: For $x \in E$ and y'', fixed, let (y_α) be a net in E tending weak-star to y''. Since the function $y \to A(x, y)$ is in E', it follows that $\lim_\alpha A(x, y_\alpha) \equiv \hat{A}(x, y'')$ exists. Next, for fixed $x'' \in E''$, take a net (x_β) in E converging weak-star to x''. Arguing in exactly the same way, we conclude that $\lim_\beta \hat{A}(x_\beta, y'') \equiv \hat{A}(x'', y'')$ exists. In fact, the value of $\hat{A}(x'', y'')$ may well depend on the order we use in our definition; that is, $\lim_\beta(\lim_\alpha A(x_\beta, y_\alpha))$ is not in general the same as $\lim_\alpha(\lim_\beta A(x_\beta, y_\alpha))$. This is related to what is often called Arens regularity [A]. On the other hand, it is clear that the two definitions coincide on the diagonal; that is, that $\hat{A}(x'', x'')$ is well-defined. Thus we can define $P : E'' \to C$ by $\hat{P}(x'') = \hat{A}(x'', x'')$. The extension of n-homogeneous polynomials, where n is arbitrary, is (hopefully) clear. It was recently shown by A.M. Davie and T. Gamelin [D-G] that if $f = \sum P_n \in H^\infty(B_E)$ then the function $\hat{f} = \sum \hat{P}_n \in H^\infty(B_{E''})$ and, in fact, $\|f\| = \|f\|$. Moreover, with a little work one can show that the extension mapping, which takes $f \in H^\infty(B_E)$ to $\hat{f} \in H^\infty(B_{E''})$ is multiplicative, linear, and continuous. Thus, to each $x'' \in B_{E''}$ we now have the right to associate a "semi-distinguished" homomorphism, $\hat{\delta}_{x''}$, which is given by $\hat{\delta}_{x''}(f) = \hat{f}(x'')$. It isn't hard to verify that $\pi(\hat{\delta}_{x''}) = x''$.

A very useful tool in studying the spectrum of $A(B_E)$ or $H^\infty(B_E)$ is to investigate the Fréchet algebra $H_b(E)$, consisting of entire functions $f : E \to C$ which are bounded on bounded subsets of E, with the corresponding defining family of norms $\|f\|_n = \sup\{|f(x)| \mid \|x\| \leq n\}$. Let $M_b(E)$ denote the set $\{\phi : H_b(E) \to C \mid \phi \text{ is a continuous homomorphism}\}$. Recall that ϕ is a continuous homomorphism if and only if there is a constant $C > 0$ and $n \in N$ such that for all $f \in H_b(E)$, $|\phi(f)| \leq C\|f\|_n$, that in fact $C = 1$ by a standard argument, and that it is unknown if every homomorphism ϕ on $H_b(E)$ is automatically continuous. In fact, the question of automatic continuity of such homomorphisms is equivalent to the well-known Michael Problem (See, for example, [M]). Note that there is a natural restriction mapping $\rho : H_b(E) \to H^\infty(B_E)$ with transpose ρ^t, say. A straightforward computation shows that ρ^t maps $M(H^\infty(B_E))$ into $M_b(E)$.

Let's define a "spectral radius" function for each $\phi \in M_b(E)$ as follows: $R(\phi) = \min\{R \mid |\phi(f)| \leq \|f\|_R \text{ for all } f \in H_b(E)\}$. The following theorem shows the relationship between the three spaces of homomorphisms studied.

Theorem

i. *$M(A(B_E))$ is homeomorphic to $\{\phi \in M_b(E) \mid R(\phi) \leq 1\}$.*

ii. $\rho^t(M(H^\infty(B_E))) = \{\phi \in M_b(E) \mid R(\phi) \leq 1\}$.

The behavior of the function $\rho^t : M(H^\infty(B_E))) \to \{\phi \in M_b(E) \mid R(\phi) \leq 1\}$ is analogous to the behavior of the canonical map $\pi : M(H^\infty(D)) \to \overline{D}$ discussed at the beginning of this talk. Specifically, ρ^t is one-to-one over the set $\phi \in M_b(E)$ such that $R(\phi) < 1$. Moreover, the set $\{\phi \mid R(\phi) < 1\}$, considered as a subset of $M(H^\infty(B_E))$, is a union of analytic discs all passing through δ_0.

To conclude, we briefly describe two recent, related articles. First, in [A-C-G-2], the authors investigate the following problem: Let Z be a dual Banach space with open unit ball B, let $A(B)$ denote the Banach algebra of analytic functions on B which are weak-star uniformly continuous on B, and let $P(B)$ denote the subalgebra of $A(B)$ consisting of uniform limits on $\overline{B}$ of weak-star continuous finite-type polynomials. When are $A(B)$ and $P(B)$ equal?

Second, in [A-Ch-L-P], the authors study closed subsets F of $A(B_{\ell_p})$ $1 \leq p \leq \infty$, which are generalized boundaries in the sense that for all $f \in A(B_{\ell_p})$, $\|f\| = \sup_{x \in F}|f(x)|$.

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