Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

Uniqueness and non uniqueness of the solutions of a mixed boundary value problem for the porous medium equation

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1 Introduction

In this note we state some of the results of [BCE] and give the idea of their proofs The details of the proofs and other related results will appear somewhere else.

The equation $u_t = \Delta(u^m)$ in $I\!R^3 \times [0,T)$, with $m > 1$, has been used as a model for the diffusion of a gas through a porous medium and has been widely studied, see for example [A] [DK] [CVW] and their references. The function $u(x,y,z,t)$ represents the density of the gas at the point (x,y,z) at the instant t. If one assumes that the initial distribution of the gas depends only on the variable x, that is, $u_0(x,y,z) = u_0(x)$,then the solution will also be independent of the variables z e y, that is, $u(x,y,z,t) = u(x,t)$. With this simplification the equation takes the form $u_t = (u^m)_{xx}$ in $I\!R \times [0,T)$.

*Partially supported by Fondecyt and Diuc

As it is well known the above stated equation does not have, in general, classical solutions. A particular type of solutions, the so called front travelling waves, can be obtained as follows. Let us look for solutions of the form $u(x,t) = f(c^2t - cx)$. By substitution in the equation and solving the resulting ordinary differential equation for f one obtains that

$$w(x,t) = [(\frac{m-1}{m})(c^2t - cx)_+]^{1/(m-1)}$$

is a weak solution of $u_t = (u^m)_{xx}$. Here $(x)_+$ denotes x if $x > 0$ and 0 if $x < 0$.

We note that in the case $c = 1$ the function w has the property that the flux trough $x = 0$, that is, $-(u^m)_x(0,t)$, is equal to the density $u(0,t)$ in $x = 0$. Therefore it provides a non trivial solution of the following problem

$$\begin{aligned} w_t &= (w^m)_{xx} \qquad \text{in } D_T = [0,+\infty) \times [0,T) \\ -(w^m)_x(0,t) &= w(0,t) \qquad \text{on } [0,T) \\ w(x,0) &= 0 \qquad \text{on } [0,+\infty) \end{aligned}$$

It seems natural now to ask the following question. Let $p \geq 1$ and consider the problem

$$\begin{aligned} u_t &= (u^m)_{xx} \qquad \text{in } D_T = [0,+\infty) \times [0,T) \\ -(u^m)_x(0,t) &= u^p(0,t) \qquad \text{on } [0,T) \qquad (1.1) \\ u(x,0) &= u_0(x) \qquad \text{on } [0,+\infty) \end{aligned}$$

Assuming that $u_0 \equiv 0$. For which values of p does the above stated problem have non negative non trivial solutions?

2 Existence

We will obtain solutions of (1.1) as the limit of solutions of the corresponding Neumann problem. So let us consider the problem

$$\begin{aligned} v_t &= (v^m)_{xx} \quad \text{in} \quad D_T = [0,+\infty) \times [0,T) \\ -(v^m)_x(0,t) &= h(t) \qquad \text{on } [0,T) \\ v(x,0) &= v_0(x) \qquad \text{on } [0,+\infty) \end{aligned} \tag{2.1}$$

where $h(t)$ is a non negative function in $L^\infty([0,T))$.

In order to prove the existence of weak solutions of (1.1) we consider the operator $N : L^\infty([0,T)) \to C([0,T))$ defined by $N(h)(t) = v(0,t)$ where $v(0,t)$ is the unique solution of (2.1) with initial condition $v_0(x) = u_0(x)$. It has been proved in [CEV] that N is continuous and compact. Moreover, as a consequence of the comparison results for the Neumann problem, it is order preserving. The same is clearly true for the operator $A(h) = (N(h))^p$. Now using the method of monotone iterations one can prove the existence of $T > 0$ so that A has a fixed point h in $C([0,T))$. Setting this fixed point as the Neumann data in (2.1) we obtain a solution of (1.1).

There are several integral comparison results for problem (2.1) that have been used by several authors to study the qualitative behaviour of the solutions of (2.1). As an example, a typical integral comparison argument, with a properly chosen travelling wave, gives the following lemma that estimates the density at the border in terms of the flux.

Lemma 2.1 *There exists a constant C, depending on m only, such that any solution $v(x,t)$ of (2.1) satisfies*

$$C\left(\int_0^t h(s)ds\right)^{2/(m+1)} \le v(0,t)t^{1/(m+1)}.$$

3 Uniqueness

It has been proved in [BCE] that if $u_0(0) > 0$ then (1.1) has a unique solution. We will discuss here the case $u_0 = 0$. The case when u_0 is not trivial but $u_0(0) = 0$ is not known to us.

Theorem 3.1 *If $u_0 = 0$ (1.1) has at least a non-negative non trivial solution if and only if $p < (m+1)/2$.*

Idea of the proof

Assume first that $p < (m+1)/2$ and that the initial condition is of the form au_0 where a is small and $u_0(0) > 0$. Let us define the function $F(t) = \int_0^t u^p(0,s)ds$. An aplication of Lemma (2.1) shows that $F(t)$ satisfies the differential inequality

$$C(F(t))^{2p/(m+1)} \le t^{p/(m+1)}F'(t)$$

Integrating this inequality we get that $F(T) \ge CT^{(m+1-p)/(m+1-2p)}$ where C is independent of a. Letting a tend to 0 we obtain a solution of (1.1) with

$u_0 = 0$ and such that

$$\int_0^T u^p(0,s)ds > 0,$$

and hence , non trivial.

In order to see the converse let $u(x,t)$ be a solution of (1.1) with $u_0 = 0$. Set $q = -(u^{(m-1)/2})_x$ and note that, under our hipotheses $q \geq 0$ and satisfies

$$\begin{aligned} q_t &= mu^{m-1}q_{xx} - \frac{4m^2}{m-1}u^{(m-1)/2}q_x q + \frac{m(m+1)}{m-1}q^3 \\ q(0,t) &= u^{p-(m-1)/2}(0,t) \\ q(x,0) &= 0 \end{aligned}$$

Since $p \geq (m+1)/2$ comparing with the explicit solution

$$r(x,t) = \frac{1}{(1 - 2\frac{m(m+1)}{m-1}t)^{1/2}}$$

we get

$$-(u^{(m-1)/2})_x \leq 2^{1/2}.$$

Therefore

$$-(u^{(m+1)/2})_x \leq 2^{1/2}\frac{m+1}{m-1}u.$$

Integrating this last inequality and using the fact that $\int_0^\infty u(y,t)dy = \int_0^t u^p(0,s)ds$ we obtain $F'(t) \geq C(F(t))^{2p/(m+1)}$ where C depens on m only. This inequality plus the fact that $F(0) = 0$ proves that $F = 0$ an hence $u = 0$. □

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