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VARIATIONAL ELLIPTIC PROBLEMS WHICH ARE NONQUADRATIC AT INFINITY

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INTRODUCTION

Let us consider the nonlinear Dirichlet problem

$$(P) \qquad -\Delta u = f(\mathrm{x}, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be a Carathéodory function with *subcritical growth*, that is,

$$| f(\mathrm{x}, s) | \leq a_0 | s |^{p-1} + b_0 \quad \forall\, s \in \mathbb{R}, \text{ a.e. } \mathrm{x} \in \Omega,$$

for some constants $a_0, b_0 > 0$, where $1 \leq p < 2N/(N-2)$ if $N \geq 3$ and $1 \leq p < \infty$ if $N = 1, 2$. Then, it is well-known that the weak solutions $u \in H_0^1(\Omega)$ of (P) are the critical points of the C^1 functional

$$J(u) = \frac{1}{2}\int_\Omega | \nabla u |^2 \, dx - \int_\Omega F(\mathrm{x}, u) dx \ ,$$

where $F(x, s) = \int_0^s f(x, t) dt$.

In this variational setting, the literature usually distinguishes between two situations for problem (P): the *subquadratic* situation, where the potential F satisfies $\limsup_{|s|\to\infty} F(x, s)/s^2 \leq c < \infty$, and the *superquadratic* situation, where F satisfies $\lim_{|s|\to\infty} F(\mathrm{x}, s)/s^2 = \infty$. The main goal of this paper is to present a unified approach to both situations by means of a condition of *nonquadraticity at infinity* on F. More precisely, our hypotheses are as follows:

$$(F_1)_q \qquad \limsup_{|s|\to\infty} \frac{F(\mathrm{x}, s)}{| s |^q} \leq b < \infty \quad \text{uniformly for } a.\ e.\ x \in \Omega,$$

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$$(F_2)_\mu \qquad \begin{cases} \liminf\limits_{|s|\to\infty} \dfrac{f(\mathrm{x},s)s - 2F(\mathrm{x},s)}{|s|^\mu} \geq a > 0 \text{ uniformly for } a.\ e.\ x \in \Omega \\ or \\ \limsup\limits_{|s|\to\infty} \dfrac{f(\mathrm{x},s)s - 2F(\mathrm{x},s)}{|s|^\mu} \leq -a < 0 \text{ uniformly for } a.\ e.\ x \in \Omega\ . \end{cases}$$

It should be noted that $(F_1)_q$ is always satisfied with $q = p$. However, it may be satisfied for smaller values of q and this will be of interest for what we intend to do.

Now, hypothesis $(F_2)_\mu$ is the one which (for heuristic reasons) we call the condition of *nonquadraticity at infinity* for the potential F. Such a condition was introduced in [4,5] with $\mu = 1$ to treat subquadratic elliptic systems and, in particular, a large class of *resonant* problems. In order to illustrate its meaning, let us first suppose that $f(\mathrm{x}, s)$ is of the form $f(\mathrm{x}, s) = \lambda s + g(s)$ for some $\lambda \in I\!R$. Then, assuming that g satisfies the *Landesman-Lazer type* condition (see [6])

$$(LL) \qquad \lim_{s\to\pm\infty} g(s) = g_\pm, \text{ with } \pm g_\pm > 0,$$

it follows that $|s|^{-1} G(s) \longrightarrow |g_\pm|$ as $s \to \pm\infty$ so that F satisfies $(F_2)_\mu$ for any $\mu \leq 1$.

On the other hand, let us now consider the well-known condition

$$(AR) \qquad 0 < \theta F(\mathrm{x}, s) \leq f(x, s)s\ ,\quad \forall\, |s| \geq R, \text{ uniformly for } a.\ e.\ x \in \Omega\ ,$$

for some $\theta > 2$ and $R > 0$, introduced by Ambrosetti-Rabinowitz [1] in order to obtain a nontrivial solution in the superquadratic case. In fact, this condition implies that $F(\mathrm{x}, s) \geq a_1 |s|^\theta \ \ \forall\, |s| \geq M$, for some $a_1 > 0$, hence

$$\frac{f(\mathrm{x},s)s - 2F(\mathrm{x},s)}{|s|^\mu} \geq (\theta - 2)\frac{F(\mathrm{x},s)}{|s|^\mu} \geq (\theta - 2)a_1 |s|^{\theta-\mu}\ ,$$

and it follows that F satisfies $(F_2)_\mu$ for any $\mu \leq \theta$.

We also recall that either (LL) or (AR) yields the *Palais-Smale* compactness condition needed is the classical variational methods. Here we show that $(F_1)_q$ and $(F_2)_\mu$ (with an additional technical restriction on the values of μ) imply a compactness condition introduced by Cerami in [3], and used by Bartolo-Benci-Fortunato in [2] to prove a general minimax theorem (Thm 2.3 in [2]) from which our results will follow. In order to get the *linking* needed in such a theorem, we will assume the following *crossing* of the first eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$:

$$(F_3) \quad \limsup_{s\to 0} \frac{2F(\mathrm{x},s)}{s^2} \leq \alpha < \lambda_1 < \beta \leq \liminf_{|s|\to\infty} \frac{2F(\mathrm{x},s)}{s^2}, \text{ uniformly for } a.\ e.\ x \in \Omega\ ,$$

Note that the left half of (F_3) implies $f(\mathrm{x}, 0) \equiv 0$, so that problem (P) has the trivial solution in this case. Now we are ready to state our main result.

Theorem Assume that F satisfies $(F_1)_q$, $(F_2)_\mu$ and (F_3) with $\mu > \frac{N}{2}(q-2)$. Then problem (P) has a nontrivial solution $u \in H_0^1(\Omega)$.

It is clear that the above result allows a unified treatment of subquadratic and superquadratic variational problems. Moreover, we are able to treat problems which (although subquadratic at the level of the potential $F(x,s)$) are in the interface between the sublinear and superlinear situations, at the level of the nonlinearity $f(x,s)$ itself.

In the next section we present the preliminary results which are needed in our treatment of problem (P) and we sketch the proof of the above theorem. Detailed proofs and other extensions will appear elsewhere.

PROOF OF THE MAIN RESULT

Let us denote by $\|\cdot\|$ the norm in $H_0^1(\Omega)$ induced by the inner product

$$\langle u,v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx \ , \quad u,v \in H_0^1(\Omega) \ .$$

We start recalling a *compactness condition* of the *Palais-Smale type* which was introduced by Cerami in [3] and which allows rather general minimax results (cf. [2,3]).

A functional $J \in C^1(X, \mathbb{R})$, X a real Banach space, is said to satisfy *condition (C) at the level* $c \in \mathbb{R}$ if the following holds:

$(C)_c$ (i) Any bounded sequence $(u_n) \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ possesses a convergent subsequence;

(ii) There exist constants $\delta, R, \alpha > 0$ such that $\|J'(u)\| \, \|u\| \geq \alpha$ for any $u \in J^{-1}([c-\delta, c+\delta])$ with $\|u\| \geq R$.

Remarks 1) The above condition is clearly implied by the usual *Palais-Smale condition at the level* $c \in \mathbb{R}$:

$(PS)_c$ Any sequence $(u_n) \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ possesses a convergent subsequence.

2) In the case of our problem (P), the subcritical growth condition on f automatically gives $(C)_c(i)$ $\forall c \in \mathbb{R}$.

It was shown in [2] that condition (C) actually suffices to get a *deformation theorem* and then, by standard minimax arguments, the following result was proved:

Theorem 0([3,Thm 2.3]) Suppose that $J \in C^1(H, \mathbb{R})$, H a real Hilbert space, satisfies condition $(C)_c$ $\forall c > 0$ and that there exist a closed subset $S \subset H$ and a Hilbert manifold $Q \subset H$ with boundary ∂Q verifying the following conditions:

(*a*) $\sup_{u \in \partial Q} J(u) \leq \alpha < \beta \leq \inf_{u \in S} J(u)$ for some $0 \leq \alpha < \beta$;

(*b*) S and ∂Q *link*;

(*c*) $\sup_{u \in Q} J(u) < +\infty$.

Then J possesses a critical value $c \geq \beta$.

Now, let us sketch the proof of our main result. It is based on Theorem 0 above and on the following two auxiliary results.

Lemma 1 If F satisfies $(F_1)_q$ and $(F_2)_\mu$ with $\mu > \frac{N}{2}(q-2)$ then J satisfies $(C)_c$ for every $c \in \mathbb{R}$.

Lemma 2 If (F_3) holds then there exist ρ ,$\beta > 0$ such that $J(u) \geq \beta$ if $\|u\| = \rho$. Moreover, there exists $u_0 \in H_0^1(\Omega)$ such that $J(tu_0) \longrightarrow -\infty$ as $t \to \infty$.

In view of Lemmas 1 and 2, we may apply Theorem 0 taking $S = \{u \mid \|u\| = \rho\}$ and $Q = \{tu_0 \mid 0 \leq t \leq t_0\}$, with $t_0 > 0$ being such that $J(tu_0) \leq 0$. It follows that the functional J has a critical value $c \geq \beta > 0$ and, hence, that problem (P) has a nontrivial solution $u \in H_0^1(\Omega)$.

Remark It should be mentioned that condition (F_3) naturally gives the geometry of the Mountain-Pass Theorem ([1,7]). So, Theorem 0 as used above proof is precisely that Theorem with condition (C). On the other hand, it is not hard to see that (F_3) can be replaced by other suitable *crossing of eigenvalues* which will yield *higher linkings* for the functional J and, consequently, similar existence results for problem (P).

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