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POLYNOMIALLY HYPONORMAL OPERATORS ON HILBERT SPACE [1,2]

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Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be ***normal*** if $T^*T = TT^*$, ***hyponormal*** if $T^*T \geq TT^*$, and ***subnormal*** if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal, then T is also hyponormal: For, if $N = \begin{bmatrix} T & A \\ 0 & B \end{bmatrix}$ is a normal extension of T, we have

$$0 = N^*N - NN^* = \begin{bmatrix} T^* & 0 \\ A^* & B^* \end{bmatrix} \begin{bmatrix} T & A \\ 0 & B \end{bmatrix} - \begin{bmatrix} T & A \\ 0 & B \end{bmatrix} \begin{bmatrix} T^* & 0 \\ A^* & B^* \end{bmatrix}$$

$$= \begin{bmatrix} T^*T - TT^* - AA^* & * \\ * & * \end{bmatrix},$$

so that $T^*T - TT^* = AA^* \geq 0$. The converse is false, although examples are not entirely easy to construct.

The notions of hyponormality and subnormality were introduced by P.R. Halmos in the early 50's [Hal1]. Back then, two classes of operators had very well established theories, the class of compact operators and the class of normal operators. Compact operators are norm–limits of finite–rank operators (and they could therefore be studied

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using the tools of linear algebra and basic Banach algebra theory), while normal operators were described by the powerful and deep Spectral Theorem. Thus, Halmos thought that the structure of classes slightly larger than that of normal operators could be described with the help of the Spectral Theorem, with some new additional ingredients. Forty years later, and despite the efforts of hundreds of researchers (which include developments of great depth, whose discussion would take us too far afield), both the class of subnormal operators and that of hyponormal ones still offer many undiscovered mysteries (see [Cla], [Con], [MaP], [Ptn]).

On one hand, hyponormality reflects the geometric nature of of the notion of normality, with the corresponding implications in terms of matricial positivity; on the other hand, subnormality is intimately related to the notion of analyticity for complex functions, through the restriction of the functional calculus to invariant subspaces. For the construction of models, hyponormality needs singular integrals and multiplication operators on Sobolev spaces, subnormality requires Cauchy transforms and complex function theory. Subnormality does imply hyponormality, but the significant distance between the two notions is precisely what has caused the two theories to follow separate courses. One way to compare both notions is to say that subnormality is to hyponormality as the theory of von Neumann algebras is to C^*–algebra theory.

Subnormality is invariant under polynomial calculus (since for a polynomial p, $p(S) = p(N)|_{\mathscr{H}}$, and $p(N)$ is still normal), but the square of a hyponormal operator may not be hyponormal. It is then natural to consider the class of ***polynomially hyponormal*** operators (those operators which remain hyponormal under polynomial calculus), which obviously contains all subnormal operators. Whether these two classes are the same remained unknown for over thirty–five years, and it constitutes the central problem of these notes.

Fundamental Problem. Must a polynomially hyponormal operator be necessarily subnormal?

It is not known who posed this problem first, or when, but we do know that during the early 60's the problem circulated informally in operator theory gatherings, and that towards the end of that decade, it had already appeared in the literature, in various forms. In his famous monograph, Halmos asks for a hyponormal operator whose square is not hyponormal [Hal3, Problem 209], and he adds "This is not easy. It is, in fact, bound to be at least as difficult as the construction of a hyponormal operator that is not subnormal (Problem 203), since any solution of Problem 209 is automatically a solution of Problem 203." At the beginning of the 70's, the complexity of the problem starts to surface, through the work of Abrahamse [Abr], Fan [Fan], Joshi [Jos1,2], Lubin [Lub1,2,3], Putnam [Put], Shields [Shi] and Stampfli [Sta], and later in the work of Athavale [Ath], Conway and Szymanski [CoS], and McCullough and Paulsen [McCP]. However, it is not until recently that a frontal attack on the problem is launched.

Two important events facilitate the task. On one hand, the great advances in the theory of dilations and extensions for operators, carried out by Agler, Arveson, Choi, Effros, Haagerup, Paulsen, Power, and others, make it possible to formulate the problem in terms of separation of cones of continuous functions defined on line intervals or on domains in the complex plane, especially in terms of separation of cones of polynomials; on the other hand, the availability of symbolic manipulation makes it feasible to verify and/or obtain new examples, relations, subclasses, et cetera, thus reducing the gap between subnormality and hyponormality. Nowadays, there exists a discrete bridge starting at hyponormality, and arriving, after a countable number of steps, to subnormality; there is also a similar bridge between hyponormality and polynomial hyponormality. One of the great challenges of the present time is to understand exactly how these two bridges are related.

When $\mathscr{H}$ is finite dimensional, a hyponormal operator is automatically normal; for, if T is hyponormal, then $T^*T - TT^* \geq 0$ and

$$\mathrm{trace}(T^*T - TT^*) = \mathrm{trace}(T^*T) - \mathrm{trace}(TT^*) = 0,$$

which forces T to be normal. Thus, the Fundamental Problem is intrinsically an infinite-dimensional problem, and this naturally leads to the consideration of the class of unilateral weighted shifts (which have been very well studied through the years [Shi]) as a source of examples. An important discovery was made by S. McCullough and V. Paulsen in 1988 [McCP], when they determined that in order to solve the Fundamental Problem, it sufficient to consider the class of weighted shifts.

Also in 1988, a crucial connection between the Bram–Halmos criterion for subnormality and Berger's Theorem for weighted shifts was brought to light. In the 50's, J. Bram and Halmos proved that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$ ([Bra], [Con, III.1.9]). Using the Choleski Algorithm for operator matrices, it is easy to see that this is equivalent to the positivity of the matrices $(T^{*j}T^i - T^iT^{*j})_{i,j=1}^k$, for $k = 1, 2, \ldots$. If we denote by $[A,B] := AB - BA$ the commutator of two operators A and B, and if we define T to be k–***hyponormal*** whenever the $k \times k$ matrix $M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$ is positive, then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \geq 1$ ([CMX]).

To explain C. Berger's characterization of subnormality for unilateral weighted shifts we need to introduce some notation. Given a sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \ldots$ (called the ***weights***), we denote by W_α the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that it is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The ***moments*** of α are defined by $\gamma_0 := 1$, $\gamma_{n+1} := \alpha_n^2 \gamma_n$ $(n \geq 0)$. Berger's Theorem states that W_α is subnormal if and only if the sequence $\{\gamma_n\}$ can be interpolated by a probability measure μ supported on the interval $[0, \sup_n |\alpha_n|]$, i.e.,

$$\gamma_n = \int t^n \, d\mu(t) \qquad (\text{all } n \geq 0)$$

([Con, III.8.16], [Hal2]); briefly said, W_α is subnormal if and only if the moments of α are the moments of some probability measure μ. This immediately establishes a connection between unilateral weighted shifts and the classical theory of moments, which has been quite useful. On one hand, all the tools and techniques available from the work of

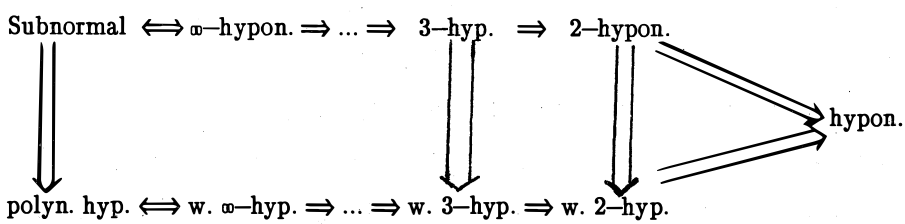
Hamburger, Stieltjes, Hausdorff, Nevanlinna, Krein, Nudel′man, Shohat, Tamarkin, and many others (see [AhK], [Akh], [KrN], [Sar], [ShT], [Sto]), can be applied to the theory of weighted shifts; on the other hand, results from operator theory can be employed to give new interpretations of moment problems, or to obtain matricial variants of them ([AtP], [Atz], [BeM], [Cas], [CuF2], [Emb], [Lam], [McG], [Nar], [Sch], [StSz], [Tre]).

Typical examples of subnormal shifts are the (***un–weighted***) unilateral shift $U_+ e_n := e_{n+1}$, the ***Bergman*** shift $B_+ e_n := (\frac{n+1}{n+2})^{1/2} e_{n+1}$, and the ***flat*** shift, whose weight sequence is a, 1, 1, 1, ... $(0 < a < 1)$. (The associated measures for the three shifts are $\mu = \delta_1$, $d\mu(t) = dt$, and $\mu = (1-a^2)\delta_0 + a^2\delta_1$, respectively.) On the contrary, the shift whose weights are a, b, 1, 1, 1, ... $(0 < a < b < 1)$ is not subnormal.

While subnormality is related to a moment problem, k–hyponormality for weighted shifts admits a matricial characterization, as follows: W_α is k–hyponormal if and only if $(\gamma_{n+i+j})_{i,j=0}^k$ is positive for all $n \geq 0$ ([Cu2,3]). When combined with the Bram–Halmos criterion, one obtains that W_α is subnormal if and only if the matrices $(\gamma_{n+i+j})_{i,j=0}^k$ are positive for all $n \geq 0$ and for all $k \geq 1$, which can be seen to be equivalent to the positivity of the two infinite matrices $(\gamma_{i+j})_{i,j=0}^\infty$ and $(\gamma_{i+j+1})_{i,j=0}^\infty$. By the classical result of Stieltjes, this in turn is equivalent to the existence of a probability measure supported on $[0,+\infty)$ which interpolates the sequence $\{\gamma_n\}$. This provides a new proof of Berger′s Theorem. The matricial criterion for k–hyponormality also provides a technique for distinguishing between k–hyponormality and (k+1)–hyponormality, and it is particularly useful in the study of recursively generated weighted shifts ([CuF1,2]). Another consequence is that we now have a three–way link among operator theory, matrix theory and measure theory, as follows: a subnormal shift corresponds to two positive Hankel matrices, which in turn correspond to a compactly supported measure on $[0,+\infty)$. Thus, results in any of these three areas must admit analogues in the other two. For instance, we know that if instead of the two matrices $(\gamma_{i+j})_{i,j=0}^\infty$ and $(\gamma_{i+j+1})_{i,j=0}^\infty$ we postulate only the positivity of the first one, for the resulting measure μ we can only assert that its support is in $(-\infty,+\infty)$ (Hamburger′s moment problem).

<u>**Problem 1**</u>. What is the operator–theoretic notion that goes with measures supported in $(-\infty,+\infty)$, along the lines of Berger′s Theorem.

Although k–hyponormality for weighted shifts admits a nice characterization, the same cannot be said of polynomial hyponormality. Despite the fact that the latter can also be interpreted as a limiting case of the notion of ***weak*** k–hyponormality (where the positivity of the matrix $([T^{*j},T^i])_{i,j=1}^k$ is replaced by its weak positivity, i.e., by the condition that $([T^{*j},T^i])_{i,j=1}^k$ be positive on vectors of the form $(\lambda_1 x,\dots,\lambda_k x)$, $\lambda_1,\dots,\lambda_k \in \mathbb{C}$, $x \in \mathcal{H}$), no significant description of polynomial hyponormality is available at present. Only quadratic hyponormality (which corresponds to weak 2–hyponormality) has lent itself to a detailed analysis, because of the existence of a peculiar characterization in terms of recursively given determinants. We'll discuss this characterization a bit later, but first we want to pause to indicate how the notions introduced so far are related:



When W_α is hyponormal, each upper–left–hand corner of the infinite matrix $[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]$ is given as

$$P_n[(W_\alpha+sW_\alpha^2)^*,W_\alpha+sW_\alpha^2]P_n = \begin{bmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{bmatrix}$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0,\dots,e_n\}$,

$$q_n = u_n + |s|^2 v_n, \qquad r_n = s\sqrt{w_n},$$

$$u_n := \alpha_n^2 - \alpha_{n-1}^2,\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2,$$

$$w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2,$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} := 0$. The determinant d_n of the tri–diagonal matrix above satisfies the following 2–step recursive formula:

$$d_0 = q_0$$
$$d_1 = q_0q_1 - |r_0|^2$$
$$d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2 d_n;$$

if we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write

$$d_n = \sum_{i=0}^{n+1} c(n,i)t^i,$$

then the coefficients $c(n,i)$ satisfy a double–indexed recursive formula, namely

$$(*)\quad \begin{aligned} &c(n+2,i) = u_{n+2}c(n+1,i) + v_{n+2}c(n+1,i-1) - w_{n+1}c(n,i-1) \\ &c(n,0) = u_0\cdot\ldots\cdot u_n,\ c(n,n+1) = v_0\cdot\ldots\cdot v_n,\ c(1,1) = u_1v_0 + v_1u_0 - w_0 \end{aligned}$$

$(n \geq 0, i \geq 1)$. Using $(*)$ in a judicious manner, one can obtain a number of results about quadratic hyponormality, which we shall explain in a moment. We want to pause, however, to mention that the only known technique to verify the quadratic hyponormality of an operator (other than showing that it belongs to a more restrictive class) is to check that all the coefficients $c(n,i)$ are non–negative. (When this is the case, $d_n(t) > 0$ for all $t > 0$, and Choleski's Algorithm implies that $[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2] \geq 0$.) In particular, the following question remains open.

__Problem 2__. Does there exists a quadratically hyponormal weighted shift with at least one $c(n,i)$ negative?

To discuss the usefulness of $(*)$, let us first recall an old result of J. Stampfli [Sta]: If $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots$, and $\alpha_n = \alpha_{n+1}$ for some n, then W_α is subnormal if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \ldots$, i.e., if and only if W_α is flat. In an effort to start unraveling the structure of quadratically hyponormal and 2–hyponormal shifts, we can ask: Are there similar results for these classes?

For 2–hyponormality, we can use the matricial characterization mentioned before to see at once that when W_α is hyponormal and $\alpha_n = \alpha_{n+1}$ for some n, then W_α is 2–hyponormal if and only if W_α is flat, which shows that Stampfli's result really pertains to the class of 2–hyponormal shifts. Concerning weak 2–hyponormality, A. Joshi proved in 1971 that the shift $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \cdots = b$, $0 < a < b$, is **not** quadratically hyponormal [Jos1,2], and later P. Fan established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + sW_\alpha^2$ is **not** hyponormal. With the aid of symbolic manipulation, and the recursive relations for d_n, it was shown in [Cu2,3] that a hyponormal weighted shift with **three** equal weights can't be quadratically hyponormal without being flat. A natural question then arises: Can a quadratically hyponormal shift have two equal weights without being flat?

The existence of such shifts was established in [Cu2], and it led to an essential distinction between 2–hyponormality and quadratic hyponormality, which eventually became the starting point for an inductive procedure to separate subnormality from polynomial hyponormality. Surprisingly enough, such examples can be constructed by considering suitable rank–one perturbation of the Bergman shifts. For $x > 0$, let W_x be the weighted shift with weights $\alpha_0 = x$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}$, $\alpha_3 = \sqrt{\frac{4}{5}}, \ldots$ $(x > 0)$. W_x is a close relative of the Bergman shift B_+ (obtained when $x = \sqrt{\frac{1}{2}}$). As shown in [Cu2],

$$W_x \text{ is subnormal} \Longleftrightarrow 0 < x \le \sqrt{\tfrac{1}{2}},$$

$$W_x \text{ is 2–hyponormal} \Longleftrightarrow 0 < x \le \tfrac{3}{4},$$

and, more generally, there exists a sequence $\{\lambda_k\}_{k=1}^{\infty}$ of positive numbers such that

$$\lambda_k > \lambda_{k+1} \quad \text{for all } k \ge 1,$$

$$\lim_k \lambda_k = \sqrt{\tfrac{1}{2}},$$

and

$$W_x \text{ is } k\text{–hyponormal} \Longleftrightarrow 0 < x \le \lambda_k, \quad (k \ge 1).$$

Also,

$$W_x \text{ is quadratically hyponormal} \Longleftrightarrow 0 < x \leq \sqrt{\tfrac{2}{3}}.$$

As a direct consequence of this, we see that the shift with weights

$$\alpha_0 = \sqrt{\tfrac{2}{3}},\ \alpha_1 = \sqrt{\tfrac{2}{3}},\ \alpha_2 = \sqrt{\tfrac{3}{4}},\ \alpha_3 = \sqrt{\tfrac{4}{5}}, \ldots$$

is quadratically hyponormal. The 4–decimal expressions for the specific values for 2–, 3–, 4–, and 5–hyponormality are .7500, .7303, .7217 and .7171, respectively, and for quadratic hyponormality .8165; however, it is not yet known what the value for **cubic** hyponormality is. Part of the difficulty is related to the fact that the corresponding recursive algorithm to compute determinants for penta–diagonal matrices consists of five steps (as opposed to two for quadratic hyponormality); the other major difficulty is the presence of two parameters, u and v, associated with the self–commutator of $W_x+uW_x^2+vW_x^3$. In analogy with the definition of λ_k above, let $\tilde{\lambda}_k$ denote the "modulus of weak k–hyponormality" $(k \geq 1)$.

__Problem 3__. Calculate explicitly the value of $\tilde{\lambda}_3$.

For higher values of k, the detection of weak k–hyponormality becomes quite hard, mainly because no recursive algorithm for calculating determinants of hepta–diagonal matrices (and nona–diagonal, etc.) is known. However, in the case at hand, it would suffice to check whether $\lim_k \tilde{\lambda}_k > \sqrt{\tfrac{1}{2}}$ to produce a **concrete example** of a non–subnormal polynomially hyponormal shift. This, however, seems to be quite inaccessible at present.

Connected with the above example is the problem of finding adequate descriptions of quadratic hyponormality. For instance, one would like to parameterize all quadratically hyponormal shifts whose first two weights are equal to 1. Symbolic manipulation shows that there are no such shifts with $1 < \alpha_2 = \alpha_3$, that α_2 is always less than $\sqrt{2}$, and that $\alpha_3 \geq (2-\alpha_2^2)^{-\frac{1}{2}}$.

Problem 4. Describe all quadratically hyponormal shifts with $\alpha_0 = \alpha_1 = 1$.

There is another class of unilateral weighted shifts that has played a key role in the recent solution of the Fundamental Problem. We are referring to the class of recursively generated weighted shifts, especially those known as "abc" type, which we now proceed to describe. Back in 1966, Stampfli showed that for arbitrary $\alpha_0 < \alpha_1 < \alpha_2$, there always exists a subnormal unilateral weighted shift T whose first **three** weights are $\alpha_0, \alpha_1, \alpha_2$; he also proved that given **four** or more weights, it may not be possible to find a subnormal completion. Stampfli's proofs were of "geometric" nature, in the sense that he built the normal extension directly out of the weights α_0, α_1 and α_2; the procedure also allowed him to conclude that four arbitrary weights cannot be subnormally completed, in general.

In search for an explanation of this phenomenon, one is naturally led to the following problem.

Subnormal Completion Problem. Given an ***initial segment*** of weights $\alpha : \alpha_0,\dots,\alpha_m$, find necessary and sufficient conditions for the existence of $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\alpha \subset \hat{\alpha}$ and $W_{\hat{\alpha}}$ is subnormal. Equivalently, find necessary and sufficient conditions for the existence of a compactly supported probability measure μ on $[0, +\infty)$ which interpolates $\gamma_0,\dots,\gamma_{m+1}$, i.e.,

$$\int t^n \, d\mu(t) = \gamma_n \quad (0 \le n \le m+1).$$

The answer is surprisingly simple, and it involves the positivity of two Hankel matrices. The Subnormal Completion Criterion, obtained in [CuF1], distinguishes two cases, according to whether m is even or odd. In the former case, say $m = 2k$, there exists a subnormal completion if and only if

$$A(k) := \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{bmatrix} \ge 0,$$

$$B(k) := \begin{bmatrix} \gamma_1 & \cdots \gamma_k & \gamma_{k+1} \\ \gamma_2 & \cdots \gamma_{k+1} & \gamma_{k+2} \\ \vdots & \vdots & \vdots \\ \gamma_{k+1} & \cdots \gamma_{2k} & \gamma_{2k+1} \end{bmatrix} \geq 0,$$

and the vector

$$v(k+1,k) := \begin{bmatrix} \gamma_{k+1} \\ \gamma_{k+2} \\ \vdots \\ \gamma_{2k+1} \end{bmatrix}$$

belongs to the range of the matrix $A(k)$. When $m = 2k-1$, the criterion requires that the matrix $A(k)$ be positive, that

$$B(k-1) := \begin{bmatrix} \gamma_1 & \cdots & \gamma_k \\ \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & & \vdots \\ \gamma_k & \cdots & \gamma_{2k-1} \end{bmatrix} \geq 0,$$

and that the vector

$$v(k+1,k-1) := \begin{bmatrix} \gamma_{k+1} \\ \gamma_{k+2} \\ \vdots \\ \gamma_{2k} \end{bmatrix}$$

belong to the range of $B(k-1)$. Along the course of the proof one discovers that when $\alpha : \alpha_0, \ldots, \alpha_m$ admits a subnormal completion, then it admits one whose associated measure μ is finitely atomic. If i is the rank of $A(k)$, then it is possible to obtain an **extremal** μ (with exactly i atoms, whose moments are minimal in the class of finitely atomic interpolating measures, and which generates a shift completion of minimum norm); this μ gives rise to a ***recursive*** completion, in the sense that there exist scalars $\varphi_0, \ldots, \varphi_{i-1}$ such that

$$\gamma_{n+i} = \varphi_0\gamma_n + \dots + \varphi_{i-1}\gamma_{n+i-1} \quad \text{for all } n \geq 0,$$

or, at the level of the weights,

$$\alpha^2_{2k+j} := \varphi_{i-1} + \frac{\varphi_{i-2}}{\alpha^2_{2k+j-1}} + \dots + \frac{\varphi_0}{\alpha^2_{2k+j-1} \cdot \dots \cdot \alpha^2_{2k+j-i+1}} \quad (j \geq 1).$$

A special case is obtained when $m = 2$, and it leads to the "abc" type mentioned before. For simplicity, assume further that $\alpha_0 < \alpha_1 < \alpha_2$, and let $a := \alpha_0^2$, $b := \alpha_1^2$ and $c := \alpha_2^2$. Here

$$A(1) = \begin{bmatrix} 1 & a \\ a & ab \end{bmatrix}, \quad B(1) = \begin{bmatrix} a & ab \\ ab & abc \end{bmatrix} \quad \text{and} \quad \mathbf{v}(1,2) = \begin{bmatrix} ab \\ abc \end{bmatrix},$$

and since $\det A(1) = a(b-a) > 0$, we see at once that a subnormal completion always exist (this provides a new and simple proof of Stampfli's result). The recursion coefficients are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \quad \text{and} \quad \varphi_1 = \frac{b(c-a)}{b-a},$$

and the extremal interpolating measure is $\mu = \rho_0 s_0 + \rho_1 s_1$, where s_0 and s_1 are the two different roots of the equation $t^2 - \varphi_1 t - \varphi_0 = 0$, and ρ_0 and ρ_1 are the unique solutions of the Vandermonde system

$$\begin{bmatrix} 1 & 1 \\ s_0 & s_1 \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}.$$

For instance, if $\alpha_0 := 1$, $\alpha_1 := \sqrt{2}$ and $\alpha_2 := \sqrt{3}$, we get $\varphi_0 = -2$, $\varphi_1 = 4$, and

$$\alpha^2_{n+1} = 4 - \frac{2}{\alpha_n^2}.$$

The corresponding Berger measure is $\mu = \rho_0 s_0 + \rho_1 s_1$, with $\rho_0 = \frac{2+\sqrt{2}}{4}$, $s_0 = 2 - \sqrt{2}$, $\rho_1 = \frac{2-\sqrt{2}}{4}$ and $s_1 = 2 + \sqrt{2}$.

Recursive shifts can be regarded as elementary building blocks in the study of k–hyponormality and subnormality for weighted shifts, as the following result shows:

([CuF1]) Every subnormal unilateral weighted shift is the norm–limit of recursive shifts.

Before we proceed, we would like to indicate briefly the main ideas entering into the proof of the Subnormal Completion Criterion. (For details about the actual construction of interpolating measures and for an account of their basic properties, we refer the reader to [CuF1,2].) We start with an old result of J.L. Smul′jan [Smu]:

Let

$$\tilde{A} := \begin{bmatrix} A & b \\ b^* & c \end{bmatrix}, \quad A \in M_n(\mathbb{C}),\ \ b \in \mathbb{C}^n,\ \ c \in \mathbb{C}.$$

Then

$$\tilde{A} \geq 0 \iff A \geq 0,\ \ b = Aw \ \text{ and } \ c \geq w^* A w.$$

As an immediate consequence, we see that if $A \geq 0$ and $\operatorname{rank} \tilde{A} = \operatorname{rank} A$, then $\tilde{A} \geq 0$. This simple criterion for preservation of positivity when augmenting a matrix by one row and one column is crucial for us; the same can be said of the following result of Frobenius and Gundesfinger [Ioh, Chapter I]:

If $A = (a_{ij})_{i,j=0}^{\infty}$ is positive and if $A(k) := (a_{ij})_{i,j=0}^{k}$, then

(i) $\det A(k) = 0 \Rightarrow \det A(k+1) = 0$

(ii) $\operatorname{rank} A(k+1) \leq \operatorname{rank} A(k) + 1$.

We also require a rank principle, which states that if a positive Hankel matrix $A := (\gamma_{i+j})_{i,j=0}^{\infty}$ has finite rank k, then $A(k-1) := (\gamma_{i+j})_{i,j=0}^{k-1}$ must be invertible [CuF1, Proposition 2.12]. Next, a propagation result for square matrices is needed:

For $n \geq 2$, let $C \in M_n(\mathbb{C})$, $C \geq 0$, and suppose that

$$C = \begin{bmatrix} R & * \\ * & * \end{bmatrix} = \begin{bmatrix} * & S \\ * & * \end{bmatrix},$$

where $R, S \in M_{n-1}(\mathbb{C})$. Then $\mathrm{rank}(S) \leq \mathrm{rank}(R)$.

Finally (and most importantly), we require a structure result for positive Hankel matrices which is naturally and intrinsically tied to the notion of recursiveness. It states that if $A = (\gamma_{i+j})_{i,j=0}^{n}$ is a positive Hankel matrix, and if $A(k) := (\gamma_{i+j})_{i,j=0}^{k}$ is the first compression with determinant 0, then **every** entry of A, with the possible exception of γ_{2n}, is **completely determined** by $A(k-1)$.

Our techniques are elementary and general, and they also allows us to obtain solutions to the truncated Hamburger, Hausdorff, and Toeplitz moment problems (cf. [CuF2]). Here, however, we are mainly interested in the applications to unilateral weighted shifts, which will lead us to the conclusion that the classes of quadratically hyponormal and 2–hyponormal shifts are indeed quite different.

As mentioned before, there is a simple characterization of 2–hyponormality (W_α is 2–hyponormal if and only if $(\gamma_{n+i+j})_{i,j=0}^{2} \geq 0$ for every $n \geq 0$), but the same cannot be said of quadratic hyponormality. As a matter of fact, here are two of the main problems still unresolved in this topic.

<u>Problem 5</u>. Is there a characterization of quadratic hyponormality along the lines of the above mentioned characterization of 2–hyponormality?

<u>Problem 6</u>. Find models for quadratic hyponormality.

One approach to the second problem is to think of quadratically hyponormal shifts as perturbations of subnormal shifts, and to recall that these are norm–limits of recursively generated shifts; thus, one is led to the consideration of perturbations of recursive subnormal shifts, those given by finitely atomic measures on $[0,+\infty)$. A concrete situation is the following:

Assume that $\alpha_0 < \alpha_1 < \alpha_2$ are given. For which x's is $W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ quadratically hyponormal?

(Recall that by $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ we mean the shift whose weights are calculated according to the recursive relation $\alpha_{n+1} = \varphi_1 + \frac{\varphi_0}{\alpha_n^2}$, where $\varphi_0 = -\frac{ab(c-b)}{b-a}$ and $\varphi_1 = \frac{b(c-a)}{b-a}$; $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is subnormal, and we perturb it by inserting x as the zeroth. weight.)

To start, we would like to find the range of x's for which W_x is 2–hyponormal. By a special case of the Extension Principle ([CuF1, Theorems 5.7 and 5.10]), this happens precisely when the shift $W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is subnormal, or equivalently when $x^2 \leq (\frac{\rho_0}{s_0} + \frac{\rho_1}{s_1})^{-1}$, where $\mu = \rho_0\delta_{s_0} + \rho_1\delta_{s_1}$ is the measure associated to $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$. For the example $a = 1$, $b = 2$, $c = 3$, one obtains that the corresponding shift is 2–hyponormal if and only if $x^2 \leq \frac{2}{3}$.

For quadratic hyponormality, the actual calculation of the range of x's is much more difficult, and it involves heavy use of symbolic manipulation.

Quadratic Hyponormality Criterion. ([CuF1]) Let

$$H_2 := \sup \{ x > 0: W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge} \text{ is 2–hyponormal} \},$$

$$h_2^+ := \sup \{x > 0: W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge} \text{ has } c(n,i) \geq 0, \text{ all } n,i \},$$

and write $a := \alpha_0^2$, $b := \alpha_1^2$, $c := \alpha_2^2$. Then

1) $$H_2 = \left(\frac{\varphi_0}{a-\varphi_1}\right)^{\frac{1}{2}} = \left(\frac{a^2b^2(c^2-b^2)}{(b^2-a^2)^2 + b^2(c^2-b^2)}\right)^{\frac{1}{2}};$$

2) $$h_2^+ = \left(\frac{a^2b^2c + ab^2(c-a)K + ab(c-b)K^2}{a^3b + ab(c-a)K + (a^2+bc-2ab)K^2}\right)^{\frac{1}{2}},$$

where $K := -\frac{\varphi_1^2 L^2}{\varphi_0}$, with $L := \|W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}\| = (\frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_0}}{2})^{\frac{1}{2}}$;

and

3) $H_2 < h_2^+$.

When $a = 1$, $b = 2$ and $c = 3$, we get $h_2^+ \cong 0.8563$ and $H_2 = \sqrt{\frac{2}{3}} \cong 0.8165$. Of course the most important of the three statements is the last one, since it tells us that no matter how we choose a, b and c, we can always find x's (a whole interval of them!) such that the shift $W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is quadratically hyponormal and not 2–hyponormal. Similar techniques can also be used to show that there exists $\epsilon > 0$ with the following property: For every $1 < \alpha_1 < 1 + \epsilon$ there exists $\alpha_2 > \alpha_1$ such that $W_{1\,(1,\alpha_1,\alpha_2)^\wedge}$ is quadratically hyponormal. (Recall that $W_{1\,(1,\alpha_1,\alpha_2)^\wedge}$ **can't** be 2–hyponormal.) This shows that non–trivial quadratically hyponormal shifts with two equal weights are quite common, and deserve to be fully classified.

Although the proof of the Quadratic Hyponormality Criterion is quite involved, we would like to give some idea of how symbolic manipulation was used. First, part 1) follows from the Extension Principle. As for part 2), direct calculation shows that $c(n,i) \geq 0$ for $n \leq 2$ and all i. For $n \geq 3$, and $0 \leq i \leq n+1$, the recursive formula

$$(*) \quad \begin{aligned} & c(n+2,i) = u_{n+2}c(n+1,i) + v_{n+2}c(n+1,i-1) - w_{n+1}c(n,i-1) \\ & c(n,0) = u_0 \cdot \ldots \cdot u_n,\ c(n,n+1) = v_0 \cdot \ldots \cdot v_n,\ c(1,1) = u_1v_0 + v_1u_0 - w_0 \end{aligned}$$

$(n \geq 0, i \geq 1)$ (associated to the calculation of the determinant of tri–diagonal matrices) and the recursiveness of the weights α_n readily imply that

$$c(n,i) = u_n c(n-1,i) + v_n \cdot \ldots v_3[v_2 c(1,i-n+1) - w_1 c(0,i-n+1)].$$

Let

$$\rho := v_2 c(1,1) - w_1 c(0,1) \geq 0$$

and

$$\tau := v_2c(1,0) - w_1c(0,0) \leq 0$$

An inductive argument gives $c(n,i) \geq 0$ for $n \geq 5$ and $i \neq n-1$. To handle the coefficients $c(n,n-1)$, we let $z_n := \frac{u_n}{v_n}$, prove that z_n increases to K, and that

$$R(z) := \frac{a^2b^2c + ab^2(c-a)z + ab(c-b)z^2}{a^3b + ab(c-a)z + (a^2+bc-2ab)z^2}$$

is a decreasing function of z. A careful analysis of $c(n,n-1)$ reveals that 2) holds provided one can establish that

$$x \leq (R(K))^{\frac{1}{2}} \Rightarrow c(4,3) \geq 0, \ c(3,2) \geq 0.$$

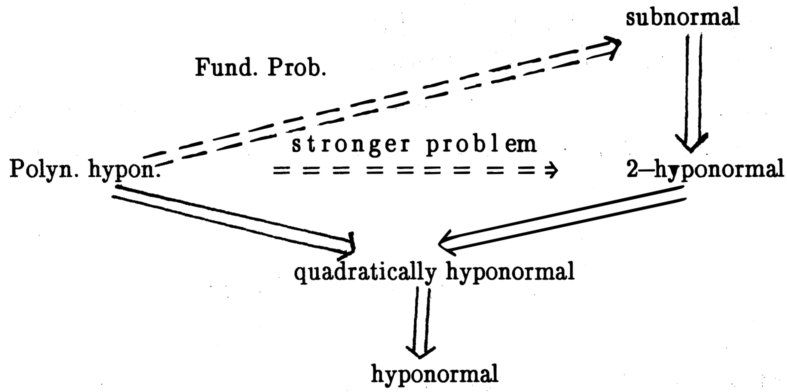
This is done as follows: We let $\delta := b-a$, $\epsilon := c-b$, $x_{3,2} := \sup \{ x > 0: \ c(3,2) \geq 0 \}$ and $x_{4,3} := \sup \{ x > 0: \ c(4,3) \geq 0 \}$, and we compute

$$\begin{aligned} x_{3,2}^2 - x_{4,3}^2 = {} & (a\delta^5 + 2a^2\delta^4 + a^3\delta^3)\epsilon^7 + (6a\delta^6 + 13a^2\delta^5 + 8a^3\delta^4 + a^4\delta^3)\epsilon^6 \\ & + (15a\delta^7 + 35a^2\delta^6 + 25a^3\delta^5 + 5a^4\delta^4)\epsilon^5 \\ & + (20a\delta^8 + 50a^2\delta^7 + 40a^3\delta^6 + 11a^4\delta^5 + a^5\delta^4)\epsilon^4 \\ & + (15a\delta^9 + 40a^2\delta^8 + 35a^3\delta^7 + 12a^4\delta^6 + 2a^5\delta^5 + a^6\delta^4)\epsilon^3 \\ & + (6a\delta^{10} + 17a^2\delta^9 + 16a^3\delta^8 + 6a^4\delta^7 + a^5\delta^6)\epsilon^2 \\ & + (a\delta^{11} + 3a^2\delta^{10} + 3a^3\delta^9 + a^4\delta^8)\epsilon, \end{aligned}$$

which shows that $x_{3,2} > x_{4,3}$. Similarly, $x_{4,3}^2 - R(K)$ can be reduced to a sum of some 240 positive terms. Finally, 3) is just a "brute force" calculation.

In addition to providing many concrete examples of non–subnormal quadratically hyponormal operators, the above criterion gives strong evidence that the classes of polynomially hyponormal and subnormal operators are different. Actually, and since

2–hyponormality and subnormality have identical moduli for $W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge}$, the results seem to indicate that perhaps something much stronger is true, namely that the classes of polynomially hyponormal and 2–hyponormal operators are different. Let's visualize this in the following diagram:



The following theorem answers the stronger problem in the negative, and it therefore solves the Fundamental Problem.

Theorem. ([CuP1,2]) There exists a polynomially hyponormal operator T which is **not** 2–hyponormal.

By combining this with the main result in [McCP] one gets at once the following result.

Corollary. There exists a unilateral weighted shift which is polynomially hyponormal and **not** subnormal.

We briefly indicate below the main technical steps leading to the solution of the Fundamental Problem; details can be found in [CuP2]. First, we recall that Agler found in 1985 a 1–1 correspondence between contractions with a cyclic vector and certain linear functionals on $\mathbb{C}[z,\bar{z}]$. If $T \in \mathcal{L}(\mathcal{H})$, $||T|| \leq 1$, $\gamma \in \mathcal{H}$, and $p \in \mathbb{C}[z,\bar{z}]$,

$p(z,\bar z) = \sum_{m,n} a_{mn} z^m \bar z^n$, we define the so–called ***hereditary functional calculus*** by letting $p(T,T^*) := \sum_{m,n} a_{mn} T^{*n} T^m$, and a linear functional $\Lambda_T : \mathbb{C}[z,\bar z] \longrightarrow \mathbb{C}$ by $\Lambda_T(p) := (p(T,T^*)\gamma,\gamma)$, $p \in \mathbb{C}[z,\bar z]$. Λ_T satisfies two important properties: (i) $\Lambda_T(p\bar p) \geq 0$, and (ii) $\Lambda_T((1 - z\bar z)p\bar p) \geq 0$. Conversely, if $\Lambda : \mathbb{C}[z,\bar z] \longrightarrow \mathbb{C}$ is a linear functional satisfying (i) and (ii), we let $\mathscr{N} := \{p \in \mathbb{C}[z]: \Lambda(p\bar p) = 0\}$, and observe that $\Lambda\mathscr{N} \subseteq \mathscr{N}$, and that $\mathbb{C}[z]/\mathscr{N}$ is a pre–Hilbert space with the inner product $\langle p,q\rangle := \Lambda(p\bar q)$, $p,q \in \mathbb{C}[z]$. Moreover, $z\mathscr{N} \subseteq \mathscr{N}$. If we now let $T := M_z$ on $\mathscr{H} := (\mathbb{C}[z]/\mathscr{N})\hat{}$, we see that T is a contraction operator with cyclic vector $1 + \mathscr{N}$.

Next, we recall that for T cyclic with vector γ,

$$T \text{ is 2–hyponormal} \iff \begin{bmatrix} I & T^* & T^{*2} \\ T & T^*T & T^{*2}T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{bmatrix} \geq 0$$

$$\iff \left(\begin{bmatrix} I & T^* & T^{*2} \\ T & T^*T & T^{*2}T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{bmatrix} \begin{bmatrix} p_0(T)\gamma \\ p_1(T)\gamma \\ p_2(T)\gamma \end{bmatrix}, \begin{bmatrix} p_0(T)\gamma \\ p_1(T)\gamma \\ p_2(T)\gamma \end{bmatrix} \right) \geq 0$$

$$\iff \left(|p_0 + p_1\bar z + p_2\bar z^2|^2(T,T^*)\gamma,\gamma \right) \geq 0$$

$$\iff \Lambda_T(|p_0 + p_1\bar z + p_2\bar z^2|^2) \geq 0 \qquad (p_0, p_1, p_2 \in \mathbb{C}[z]).$$

Similarly,

$$T \text{ is polynomially hyponormal} \iff \begin{bmatrix} I & r(T)^* \\ r(T) & r(T)^*r(T) \end{bmatrix} \geq 0$$

$$\iff \left(\begin{bmatrix} I & r(T)^* \\ r(T) & r(T)^*r(T) \end{bmatrix} \begin{bmatrix} p(T)\gamma \\ q(T)\gamma \end{bmatrix}, \begin{bmatrix} p(T)\gamma \\ q(T)\gamma \end{bmatrix} \right) \geq 0$$

$$\iff \left(|p + q\bar r|^2(T,T^*)\gamma,\gamma \right) \geq 0$$

$$\iff \Lambda_T(|p + q\bar r|^2) \geq 0 \qquad (p, q \in \mathbb{C}[z]).$$

We are thus led to consider two cones of polynomials: $\mathscr{S}^2$, generated by all polynomials of the form $(1-z\bar{z})|p|^2$ and $|p_0 + p_1\bar{z} + p_2\bar{z}^2|^2$, and $\mathscr{W}$, generated by those of the form $(1-z\bar{z})|p|^2$ and $|p + q\bar{r}|^2$. The above calculations show that T is 2–hyponormal if and only if $\Lambda_T \big|_{\mathscr{S}^2} \geq 0$, and that T is polynomially hyponormal if and only if $\Lambda_T \big|_{\mathscr{W}} \geq 0$. From this viewpoint, the Fundamental Problem will be resolved if we can accomplish the following

Goal. Find $\Lambda : \mathbb{C}[z,\bar{z}] \longrightarrow \mathbb{C}$ and $p \in \mathscr{S}^2$ such that $\Lambda \big|_{\mathscr{W}} \geq 0$ and $\Lambda(p) < 0$.

Once this linear functional has been found, we can build T in such a way that $\Lambda_T = \Lambda$. To construct Λ, we introduce some auxiliary cones. First, we shall denote by Γ the cone generated by polynomials of the form $|p + q\bar{r}|^2$, and for $m \geq 0$, we shall denote by $\mathbb{C}[z,\bar{z}]_m$ the cone of polynomials whose total degree in z and $\bar{z}$ is at most m, by $\mathbb{C}[z,\bar{z}]^h$ the collection of homogeneous polynomials, and by $\mathbb{C}[z,\bar{z}]^h_m$ the set of homogeneous polynomials of degree m (with similar definitions for $\mathbb{R}[x,y]_m$, $\mathbb{R}[x,y]^h$ and $\mathbb{R}[x,y]^h_m$). Finally, for K a cone in $\mathbb{C}[z,\bar{z}]$, we shall let $K_m := K \cap \mathbb{C}[z,\bar{z}]_m$, $K^h := K \cap \mathbb{C}[z,\bar{z}]^h$, and $K^h_m := K \cap \mathbb{C}[z,\bar{z}]^h_m$. Observe that Γ is smaller than $\mathscr{W}$, but easier to handle; our strategy will exploit this, together with the fact that Γ^h_4 is actually equal to $\mathscr{W}^h_4$.

The first observation is that $\{ p(z,\bar{z}) \in \mathbb{C}[z,\bar{z}]^h_m : p = \bar{p} \} = \mathbb{R}[x,y]^h_m$, via the usual identification $\frac{z+\bar{z}}{2} = x$, $\frac{z-\bar{z}}{2i} = y$, so that it suffices to construct a real linear functional Λ on $\mathbb{R}[x,y]$. Next, we recall that if E is a (real, finite dimensional) vector space, if K is a convex subset of E with $\text{int}(K) \neq \phi$, and if M is a linear manifold in E such that $M \cap \text{int}(K) = \phi$, then there exists a hyperplane $H \supseteq M$ such that $H \cap \text{int}(K) = \phi$ (cf. [CoCi, I.3.1.3]).

To build Λ, we plan to use the fact that quadratic hyponormality and 2–hyponormality are far apart, and therefore it should be possible to separate $\mathscr{S}^2$ from $\mathscr{W}_4$. For technical reasons, it is more convenient to consider Γ^h_4 first. Thus, we shall attempt to define $\Lambda^h_4 : \mathbb{R}[x,y]^h_4 \longrightarrow \mathbb{R}$ such that $\Lambda^h_4 \big|_{\Gamma^h_4} \geq 0$, $\Lambda^h_4 \big|_{\text{int}(\Gamma^h_4)} > 0$, and

$\Lambda_4^h(p) < 0$ for some $p \in \mathscr{P}^2 \cap \mathbb{R}[x,y]_4^h$. Once this is done, we shall try and extend Λ_4^h to $\mathbb{R}[x,y]_4$, then to $\mathbb{R}[x,y]_5$, to $\mathbb{R}[x,y]_6$, etc. The corresponding convex sets to be considered are Γ_4^h, $\mathscr{W}_4^h$, $\mathscr{W}_4$, $\mathscr{W}_5$, $\mathscr{W}_6$, etc. Two results are needed to make the extension strategy work: on one hand, $\mathrm{int}(\Gamma_4^h)$, $\mathrm{int}(\mathscr{W}_4^h)$, $\mathrm{int}(\mathscr{W}_4)$, $\mathrm{int}(\mathscr{W}_5)$, $\mathrm{int}(\mathscr{W}_6)$, etc., must all be nonempty; on the other hand, we must verify that $\mathscr{W}_4^h = \Gamma_4^h$. This is accomplished in Steps 1 and 3 below; Step 2 is required in the actual construction of Λ_4^h.

Step 1

(i) For $m \geq 0$, $\mathbb{R}[x,y]_m = \mathscr{W}_m - \mathscr{W}_m$ ($\Rightarrow$ $\mathrm{int}.\mathscr{W}_m \neq \phi$).

(ii) For $m \geq 0$ even, $\mathbb{R}[x,y]_m = \Gamma_m - \Gamma_m = \mathscr{W}_m - \mathscr{W}_m$.

(iii) For $m \geq 0$ even, $\mathbb{R}[x,y]_m^h = \Gamma_m^h - \Gamma_m^h$ ($\Rightarrow$ $\mathrm{int}.\Gamma_m^h \neq \phi$).

Step 2. Γ_4 is generated by all polynomials of the form

$$|c_0 + c_1 z + c_2 \bar{z} + c_3 z^2 + c_4 z\bar{z} + c_5 \bar{z}^2|^2,$$

where $c_4 = 0$ or $c_5 = 0$.

Step 3. $\mathscr{W}_4^h \subseteq \Gamma_4^h$, that is, the $(1 - |z|^2) \sum_j |s_j(z)|^2$ component of an homogeneous polynomial of total degree 4 can be eliminated.

Step 4. Define $\Lambda_4^h : \mathbb{R}[x,y]_4^h \longrightarrow \mathbb{R}$ by

$$\Lambda_4^h(z^4) = 0,\ \Lambda_4^h(z^3\bar{z}) = b,\ \Lambda_4^h(z^2\bar{z}^2) = 1,\ \Lambda_4^h(z\bar{z}^3) = b,\ \Lambda_4^h(\bar{z}^4) = 0.$$

Then

$$\begin{aligned} &\Lambda_4^h(|c_3 z^2 + c_4 z\bar{z} + c_5\bar{z}^2|^2) \\ &= \Lambda_4^h(c_3\bar{c}_3 z^2\bar{z}^2 + c_3\bar{c}_4 z^3\bar{z} + c_3\bar{c}_5 z^4 + c_4\bar{c}_3 z\bar{z}^3 + c_4\bar{c}_4 z^2\bar{z}^2 \\ &\quad + c_4\bar{c}_5 z^3\bar{z} + c_5\bar{c}_3\bar{z}^4 + c_5\bar{c}_4 z\bar{z}^3 + c_5\bar{c}_5 z^2\bar{z}^2) \\ &= \left(\begin{bmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \\ c_5 \end{bmatrix}, \begin{bmatrix} c_3 \\ c_4 \\ c_5 \end{bmatrix} \right). \end{aligned}$$

The eigenvalues of this matrix are 1, $1+\sqrt{2}b$ and $1-\sqrt{2}b$. For $\frac{\sqrt{2}}{2} < b < 1$, there is a negative eigenvalue, with eigenvector $(1,-\sqrt{2},1)$, which corresponds to the polynomial

$$p(z,\bar{z}) := |\, z^2 - \sqrt{2}z\bar{z} + \bar{z}^2 |^2 \in \mathscr{S}^2.$$

However, the compressions of the matrix to $c_4 = 0$ and to $c_5 = 0$ are positive; therefore (by Step 2) $\Lambda_4^h \big|_{\Gamma_4^h} \geq 0$, and of course $\Lambda_4^h \big|_{\mathrm{int}(\Gamma_4^h)} > 0$.

Summarizing, we have found $p \in \mathscr{S}^2$ such that $\Lambda_4^h(p) < 0$ and $\Lambda_4^h \big|_{\mathrm{int}(\mathscr{W}_4^h)} > 0$. In $E = \mathbb{R}[x,y]_4$ we let $K = \mathscr{W}_4$, $M = \ker \Lambda_4^h$ and then have $M \cap \mathrm{int}(K) = \ker \Lambda_4^h \cap \mathrm{int}(\mathscr{W}_4)$ $= \ker \Lambda_4^h \cap \mathrm{int}(\mathscr{W}_4^h) = \phi$, and $\mathrm{int}(K) = \mathrm{int}(\mathscr{W}_4) \neq \phi$, so there exists Λ_4 on $\mathbb{R}[x,y]_4$ such that $\ker \Lambda_4 \cap \mathrm{int}(\mathscr{W}_4) = \phi$. We now switch to $E = \mathbb{R}[x,y]_5$ and consider $K = \mathscr{W}_5$ and $M = \ker \Lambda_4$. Since $\ker \Lambda_4 \cap \mathrm{int}(\mathscr{W}_5) = \ker \Lambda_4 \cap \mathrm{int}(\mathscr{W}_4) = \phi$, and $\mathrm{int}(\mathscr{W}_5) \neq \phi$, we see that $\exists\, \Lambda_5$ on $\mathbb{R}[x,y]_5$ such that $\ker \Lambda_5 \cap \mathrm{int}(\mathscr{W}_5) = \phi$. Next, we consider $E = \mathbb{R}[x,y]_6$, $K = \mathscr{W}_6$ and $M = \ker \Lambda_5$, and continue this process ad infinitum. The resulting linear functional Λ has the right separation properties.

The solution of the Fundamental Problem, establishing a separation between subnormality and polynomial hyponormality, gives rise to a number of open questions and provides a new viewpoint for subnormal and hyponormal operator theory. On one hand, one can now study the class of polynomially hyponormal operators on its own, and seek to extend well–known properties of subnormal operators, or try to find useful characterizations. On the other hand, even if one were to argue that the new class remains a bit artificial (mainly because no concrete nontrivial examples exist), it is clear that its study is relevant in gaining a complete understanding of the relationships between subnormality and hyponormality. Either way, a multitude of problems arise, which we proceed to list.

Problem 7. Find a concrete example of a polynomially hyponormal operator which is not 2–hyponormal?

Problem 8. Find a concrete example of a non–subnormal polynomially hyponormal weighted shift?

We notice that the separating linear functional constructed above cannot correspond to a weighted shift, since in this case, $\Lambda(z^i\bar{z}^j)$ must equal (T^ie_0, T^je_0) and so it must be zero when $i \neq j$. One possible candidate for Problem 7 is the shift studied in [CMX] (see also [Cu2, Remark 6.3]), given by

$$\alpha_0^2 := 2, \quad \alpha_k^2 := [2(1-a^2) + a^2\cdot\begin{bmatrix}2k+2\\k+1\end{bmatrix}] \;/\; [2(1-a^2) + a^2\cdot\begin{bmatrix}2k\\k\end{bmatrix}] \quad (0 < a < 1).$$

Problem 9. Does there exist a polynomially hyponormal **weighted shift** that is not 2–hyponormal?

McCullough and Paulsen used in [McCP] a symmetrization process which does not allow one to keep track of the degree of hyponormality, so a new idea seems to be needed for Problem 9. One possibility is to try to generalize Berger's Theorem, by considering linear functionals more general than probability measures.

Problem 10. Is there an analogue of Berger's Theorem for polynomially hyponormal weighted shifts?

An important breakthrough in the theory of subnormal operators was made by S. Brown in 1978, when he showed that they all possess nontrivial invariant subspaces [Bro1] (see also [Tho]). Later, he also established that hyponormal operators with thick spectra satisfy the same property [Bro2], a fact that has been extended to other operators acting on Banach spaces (see for instance [AlC], [AlE], [Esc], [EsP]).

Problem 11. Do polynomially hyponormal operators have nontrivial invariant subspaces?

In connection with M. Putinar's model for hyponormal operators [Put1,2], one can ask the following question.

Problem 12. What special properties does the 2–subscalar model have when the operator is polynomially hyponormal?

An important invariant for operators with trace–class self–commutator is the principal function introduced by J.D. Pincus (see [Cla], [MaP], [Pin], [Xia1]); for subnormal operators, this function is integer–valued [CaP].

Problem 13. Is the principal function of a polynomially hyponormal operator with trace–class self–commutator integer–valued?

The subnormal operators with finite–rank self–commutator have recently been classified by R. Olin, J. Thomson and T.Trent [OTT], and independently by D. Xia [Xia 2]).

Problem 14. Is there an analogue classification for polynomially hyponormal operators with finite–rank self–commutator?

Our final problem deals with the notions of subnormality and hyponormality on C^*–algebras. As the reader might guess, we can define an element t of a C^*–algebra $\mathscr{A}$ to be hyponormal if $t^*t \geq tt^*$; similarly, we can use Agler's criterion for subnormality [Ag2] to say that an element $t \in \mathscr{A}$ with $||t|| \leq 1$ is subnormal if $\sum_{k=0}^{M} (-1)^k \binom{M}{k} t^{*k} t^k \geq 0$ for every $M \geq 1$. Since every C^*–algebra can be regarded as a C^*–subalgebra of $\mathscr{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, it follows that a subnormal element t is always hyponormal.

Problem 15. Find a **purely C^*–algebraic proof** (which avoids the GNS construction or similar devices) of the implication "t subnormal $\Longrightarrow$ t hyponormal."

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