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Weights for the one-sided Hardy-Littlewood Maximal Operator and a..e. Convergence

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Abstract. We show how the study of the good weights for the one-sided Hardy-Littlewood Maximal Operator, leads to the solution of some problems in the theory of a.e. convergence in ergodic theory.

1. Introduction

The aim of this note is to show the relationship between two, apparently,unrelated topics. The first one is the study of weights for the one-sided Hardy-Littlewood Maximal Operators

$$M^+f(x) \equiv sup_{h>0}\frac{1}{h}\int_x^{x+h}|f|$$

and,

$$M^-f(x) \equiv sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f|$$

The problem is to study and characterize the pairs of nonnegative functions (u, v) for which one has either the weak type condition

(W-T) $$\int_{\{M^+f>\lambda\}} u \leq \frac{C}{\lambda^p}\int |f|^p v$$

or the strong type condition

(S-T) $$\int (M^+f)^p u \leq C\int |f|^p v$$

The second problem can be stated as follows: Let (X, F, μ) be a finite measure space, and let T be a positive, linear operator defined on the set of measurable functions. Let $T_nf(x) \equiv \frac{1}{n}(f + Tf + \cdots + T^{n-1}f)(x)$. When is it true that $\lim_{n\to\infty} T_nf(x)$ exists and is finite a.e. for all f in L_p. ?

It has been known for a long time, [B] that if T is of the form $Tf(x) = f(Sx)$, where $S : X \to X$ is a measure preserving transformation (i.e. $\mu(S^{-1}(A) = \mu(A)$ for all $A \in F$), then $\lim_{n\to\infty} T_nf(x) \equiv Pf(x)$ exists and is finite a.e. for all $f \in L_p$ $1 \leq p \leq \infty$. Furthermore $Pf \in L_p$. Two natural question are:

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PROBLEM 1. *Let's assume that T is of the form $Tf(x) = f(Sx)$, where now S is not measure preserving. Under which conditions is it true that $\lim_{n\to\infty} T_n f(x) \equiv Pf(x)$ exists and is finite a.e. for all $f \in L_p$*

PROBLEM 2. *Under which extra conditions can one say that $Pf \in L_p$*

In 1946 Dundford and Miller [**DM**] proved that $\lim_{n\to\infty} T_n f \equiv Pf$ exists in the L_1 norm if, and only if,

$$\text{(D-M)} \qquad \frac{1}{n}\sum_{0}^{n-1} \mu(S^{-j}A) \leq C\mu(A),$$

furthermore, this condition implies a.e. convergence of the averages $T_n f(x)$ for all $f \in L_1(d\mu)$. Later on Ryll-Nardzewski [**RN**] proved that the averages converge a.e. to a limit Pf, and $Pf \in L_1$ if, and only if,

$$\text{(R-N)} \qquad \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mu(S^{-j}A) \leq C\mu(A),$$

where C is independent of $A \in F$ and n. This results solved Problem 2 for $p = 1$, but Problem 1 for $p = 1$, and both problems for $p > 1$, remained open questions for forty years.

2. WEIGHTS FOR ONE-SIDED HARDY-LITTLEWOOD MAXIMAL OPERATORS.

In this section we state the main results about weights for one-sided maximal functions. The proofs can be found in the following references [S] [**MR**] [**MOT**] [**MT3**]

THEOREM 1.

(1) *If $p > 1$ then (W-T) $\Leftrightarrow$ there exists C such that for almost every x and every positive h the following inequality holds:*

$$(A_p^+) \qquad \int_{x-h}^{x} u \left(\int_{x}^{x+h} v^{-\frac{1}{p-1}}\right)^{p-1} \leq Ch^p$$

(2) *If $p = 1$ then (W-T) $\Leftrightarrow$ there exists C such that for almost every x and every positive h the following inequality holds:*

$$\frac{1}{h}\int_{x-h}^{x} u \leq Cv(x),$$

which is obviously equivalent to

$$(A_1^+) \qquad M^- u(x) \leq Cv(x) \quad a.e.$$

(3) *If $p > 1$ then (S-T) $\Leftrightarrow$ There exists C such that for all intervals I*

$$(S_p^+) \qquad \int_I (\sigma\chi_I)^{+p} u \leq C\int_I \sigma,$$

(where $\sigma = v^{-\frac{1}{p-1}}$)

THEOREM 2.. *If $u = v \equiv w$ then (W-T)$\Leftrightarrow$ (S-T). Furthermore w can be factorized as $w = v_1 v_2^{p-1}$, with $v_1 \in A_1^+$ and $v_2 \in A_1^-$, and there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$*

In ergodic theory we will be dealing with an operator and its powers, for that reason we are interested in the discrete version of the results stated above. For functions defined on the integers, the corresponding one-sided maximal operators are,

$$M^+f(k) = \frac{1}{n}\sum_{i=0}^{n-1}|f(k+i)|,$$

and

$$M^-f(k) = \frac{1}{n}\sum_{i=0}^{n-1}|f(k-i)|,$$

It is clear that for this operators one can obtain results similar to Theorems 1 and 2, just by replacing integrals by sums, and intervals by sets of consecutive integers.

3. Back to ergodic theory

Let's look again at condition (D-M)

$$\text{(D-M)} \qquad \frac{1}{n}\sum_{0}^{n-1}\mu(S^{-j}A) \leq C\mu(A).$$

If μ is of the form $d\mu = wdm$ with m invariant under S, then (D-M) can be written as

$$\frac{1}{n}\sum_{j=0}^{n-1}\int_{S^{-j}A} w\,dm \leq C\int_A w\,dm.$$

If S is invertible this means that

$$\frac{1}{n}\sum_{j=0}^{n-1}\int_A w(S^{-j}x)\,dm \leq C\int_A w(x)\,dm.$$

and this implies

$$\frac{1}{n}\sum_{j=0}^{n-1} w(S^{-j}x) \leq Cw(x) \qquad a.e.$$

In terms of weights, this condition just says that the function $i \to w(S^ix)$ satisfies A_1^+ with a constant independent of x a.e.. Therefore (D-M) is really a theorem about weights and it says that, for equal weights, A_1^+ implies a.e. convergence. It seems then a good idea to look at Problems 1 and 2 from the viewpoint of weights. In order to do that, we assume for simplicity that our measure is of the form $d\mu = vdm$, where m is invariant and $0 < v(x) < \infty$ *a.e.* If we look now at Problem 1 for $p > 1$, we observe that $\lim_{n\to\infty} T^nf(x) = Pf(x)$ exist and is finite *a.e.* for all $f \in L_p$ implies, via Nikishin's Theorem, [**GR**, p.536] that there exists u such that the maximal operator

$$M^+f(x) \equiv sup_{n>0}|T_nf(x)|,$$

is of weak type (p,p) i.e.

$$\text{(W-T)} \qquad \int_{\{M^+f>\lambda\}} u \leq \frac{C}{\lambda^p} \int |f|^p v$$

If we know which are the pairs of weights (u,v) for which one has (W-T), then one may find conditions on v that will insure the existence of such u. If the condition on v is sharp enough, so that it is necessary and sufficient for the existence of such u, then we will have solved our problem. The first step is then to characterize the pairs (u,v) for which the operator M^+ is of weak type. The answer is

THEOREM 3. *[MT1] [MT2] If S is invertible the operator M^+f is of weak type (p,p) iff the functions*

$$i \to u_x(i) \equiv u(S^i x) \qquad i \to v_x(i) \equiv v(S^i x)$$

satisfy A_p^+ with a constant independent of x i.e. if

$$(A_p^+(S)) \qquad \sum_{i=0}^{n-1} u(S^i x) \left(\sum_{i=n}^{2n-1} v^{-\frac{1}{p-1}}(S^i x) \right)^{p-1} \leq C n^p \qquad p > 1,$$

or

$$(A_1^+(S)) \qquad \frac{1}{n} \sum_{i=0}^{n-1} u(S^i x) \leq C v(S^{n-1} x) \qquad p = 1.$$

This means that for invertible transformations (D-M) characterizes, not only mean convergence, but the weak type of the maximal operator as well. If we look now at condition $A_1^+(S)$ we may solve Problem 1 for $p = 1$. Let's assume that $\inf_{i>0} v(S^i x) > 0$, and let's define $u(x) = inf_{i>0} v(S^i x)$, then $\frac{1}{k}\sum_{j=0}^{k-1} u(S^j x) \leq \frac{1}{k}\sum_{j=0}^{k-1} v(S^{k-1} x) = v(S^{k-1} x)$, and the pair (u,v) satisfies $A_1^+(S)$, which means that the operator M^+ is of weak type one-one, and then a.e. convergence for all $f \in L_1(vdm)$ follows easily.

Conversely, if there is a.e convergence for all $f \in L_1(vdm)$, then, as we have said above, there exists $u > 0$ so that (u,v) satisfies $A_1^+(S)$. We claim that, on a finite measure space, this implies $\inf_{k>0} v(S^k x) > 0$. In fact by making u smaller if necessary we may assume $u \in L_1(dm)$, then for any invariant set A we have

$$\int_A \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} u(S^i x) = \int_A u,$$

and this means that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} u(S^i x)$ is positive, and now condition $A_1^+(S)$ yields $\inf_{k>0} v(S^k x) > 0$.

REMARK. *In the proof of the equivalence between $A_1^+(S)$ and weak type one needs S to be invertible, but X can be σ − finite, but in order to prove that $A_1^+(S)$ implies $\inf_k v(S^k x) > 0$, we need X to have finite measure.(There are counter examples for infinite measure.)*

We have found the answer for $p = 1$. Let's try the same method for $p > 1$. We write condition $A_p^+(S)$ $(p > 1)$ in the form

$$\left(\frac{1}{n} \sum_{i=0}^{n-1} v^{-\frac{1}{p-1}}(S^i x) \right)^{p-1} \leq \frac{C}{\frac{1}{n}\sum_{-n+1}^{0} u(S^i x)}.$$

Since $\frac{1}{n}\sum_{-n+1}^{0} u(S^i x)$ is away from zero, we obtain that if (u,v) satisfies $A_p^+(S)$ $(p>1)$, then $M^+ v^{-\frac{1}{p-1}}(x) < \infty$ a.e.. Therefore if the averages $T_n f(x)$ converge to a finite limit for all $f \in L_p(vdm)$, then Nikishin's theorem plus our characterization of weak type yield that v satisfies $M^+ v^{-\frac{1}{p-1}}(x) < \infty$ *a.e.*.

Conversely, if v satisfies $M^+ v^{-\frac{1}{p-1}}(x) < \infty$ *a.e.*, we define

$$u(x) = (M^+ v^{-\frac{1}{p-1}}(x))^{-p} v^{-\frac{1}{p-1}}(x),$$

and it is easy to check that the pair (u,v) satisfies $A_p^+(S)$, which implies weak type and therefore a.e. convergence of the averages for all $f \in L_p(vdm)$. In this way we have completely solved Problem 1.

ANSWER TO PROBLEM 1. *The averages $T_n f(x)$ converge to a finite limit for all $f \in L_p(vdm)$ if, and only if, v satisfies*

$$\text{(M-T)} \qquad M^+ v^{-\frac{1}{p-1}}(x) < \infty \quad a.e. \quad (p>1),$$

or

$$\text{(M-T)} \qquad \inf_k v(S^k x) > 0 \quad a.e. \quad (p=1)$$

Let's turn our attention to Problem 2. For $p = 1$ the answer is the result of Ryll-Nardzewski, so let's concentrate on $p > 1$. We know that if v satisfies $A_p^+(S)$ i.e

$$(A_p^+(S)) \qquad \sum_0^{n-1} v(S^i x) \left(\sum_{n+1}^{2n} v^{-\frac{1}{p-1}}(S^i x) \right)^{p-1} \leq Cn^p \qquad p>1,$$

then $M^+ f \in L_p(vdm)$ which is a stronger statement than $Pf \in L_p(vdm)$, therefore we need a condition which is stronger than (M-T) but weaker than $A_p^+(S)$. Again the theory of weights suggest the answer. Condition $A_p^+(S)$ says that the necessary, and sufficient, condition for the $sup_{n>0}|T_n f(x)|$ to be in $L_p(d\mu)$ is that

$$\sup_{n>0} \frac{1}{n} \sum_0^{n-1} u(S^i x) \left(\frac{1}{n} \sum_{n+1}^{2n} v^{-\frac{1}{p-1}}(S^i x) \right)^{p-1} \leq C.$$

Since now we want, not the $\sup_n$, but the $\lim_n$ to be in $L_p(d\mu)$, it seems natural to replace in $A_p^+(S)$ the $\sup_n$ by $\lim_n$

ANSWER TO PROBLEM 2. *[MT4] The averages $T_n f(x)$ converge a.e. to a function $Pf(x)$ for all f in $L_p(vdm)$, and $Pf \in L_p(vdm)$ if, and only if, v satisfies:*

$$(\Lambda_p^+(S)) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} v(S^i x) \left(\frac{1}{n} \sum_{i=0}^{n-1} v^{-\frac{1}{p-1}}(S^i x) \right)^{p-1} \leq C,$$

The proof does not make use of the theory of weights, but uses its methods. First of all since we are assuming that our measure $d\mu$ is finite, we have that $v \in L_1(dm)$, which means that $\lim_{n\to\infty} \frac{1}{n}\sum_0^{n-1} v(S^i x)$ exists and, as before, is not zero for a.e. x. Therefore condition $\Lambda_p^+(S)$ yields that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (v^{-\frac{1}{p-1}}(S^i x))^{p-1}$$

also exists and is finite a.e.. But, according to our previous result, this implies that the averages $T_nf(x)$ converge a.e.to a finite limit $Pf(x)$ for all $f \in L_p(vdm)$. It is then enough to prove that $P(f) \in L_p(vdm)$. If we use the usual trick in weight theory of writing the averages in the form

$$T_nf(x) = \frac{1}{n}\sum_{i=0}^{n-1} f(S^ix)v^{1/p}(S^ix)v^{-1/p}(S^ix),$$

and now apply Holder's inequality and condition $\Lambda_p^+(S)$, we get

$$(Pf(x))^p \leq \frac{\lim_n T_nf^pv(x)}{\lim_n T_nv(x)},$$

where the lim in the numerator exists because $f^pv \in L_1(dm)$. Integrating this inequality, and using the ergodic theorem for measure preserving transformations for the $L_1(dm)$ functions f^pv and v, we have

$$\int (Pf(x))^pv(x)\,dm \leq C\int \frac{\lim_n T_nf^pv(x)}{\lim_n T_nv(x)}v(x)\,dm$$
$$= \int \frac{\lim_n T_nf^pv(x)}{\lim_n T_nv(x)}\lim_n T_nv(x)\,dm = \int \lim_n T_n(f^pv)\,dm = \int f^pv\,dm,$$

and one has that $\Lambda_p^+(S)$ implies that Pf is in $L_p(vdm)$. For the converse one can use either the factorization methods of J. Luis Rubio, or pure ergodic theory (see [**MT4**] for details).

SOME OPEN PROBLEMS.

(1) *For invertible transformations it is known that $A_1^+(S)$ is equivalent to the condition of Dundford Miller* (D-M), *and both are equivalent to the weak type of the maximal operator. If S is not invertible, it is known that $A_1^+(S) \Rightarrow$ weak type $\Rightarrow$* (D-M) *but it is not true that* (D-M)*implies $A_1^+(S)$. The conjecture is that* (D-M) *is equivalent to weak type, but it is not known*

(2) *Same question for $p > 1$ i.e. Find necessary and sufficient conditions for the maximal operator to be of weak type (p,p), when S is not invertible. Again $A_p^+(S)$ implies weak type but the converse is false.*

(3) *The theory of weights has been used successfully to prove the following theorem [**MT5**]. If T is an invertible, positive, linear operator on L_p, then the maximal operator $\sup_n |T_nf(x)|$ is bounded in L_p if, and only if, the averages T_nf are uniformly bounded in L_p. The assumption of S being invertible, although crucial to the proof does not appear in the statement of the result. The question is: can the theory of weights prove this same result without assuming that S is invertible?. (Brunel [**BR**] has a result on this direction, but using completely different methods)*

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