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Complexification of Operators Between L_p-Spaces

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1. During the X ELAM I reported about various results which were obtained jointly with A. Defant in [2] and [3] concerning the following problem:
For $q, p \in [1, \infty]$ and a continuous linear operator $T : L_q^{\mathbb{R}}(\mu) \to L_p^{\mathbb{R}}(\nu)$ (arbitrary measures μ and ν) consider the complexification $T^{\mathbb{C}} : L_q^{\mathbb{C}}(\mu) \to L_p^{\mathbb{C}}(\nu)$ defined by

$$T^{\mathbb{C}}(f + ig) := Tf + iTg.$$

Obviously, $\|T^{\mathbb{C}}\| \leq 2\|T\|$ and, if T is positive, it can be seen that $\|T^{\mathbb{C}}\| = \|T\|$. But this is false in general: as an example take the Walsh-matrix

$$W := \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2_\infty \to \mathbb{R}^2_1$$

(euclidean plane with the max-norm and the sup-norm) then $\|W^{\mathbb{C}}\| \geq \sqrt{2}\,\|W\|$ can be easily verified.
Define

$$k_{q,p} := \sup \left\{ \|T^{\mathbb{C}} : L_q^{\mathbb{C}}(\mu) \to L_p^{\mathbb{C}}(\nu)\| \;\middle|\; \begin{array}{l} \|T : L_q^{\mathbb{R}}(\mu) \to L_p^{\mathbb{R}}(\nu)\| \leq 1, \\ \mu,\ \nu \text{ arbitrary measures} \end{array} \right\}$$

then simple manipulations with the triangle inequality show that
(a) $k_{q,p} \leq \sqrt{2}$ whenever $p \geq 2$
(b) $k_{q,p} \leq 2^{1/p}$ whenever $p \leq 2$.
Duality gives easily
(c) $k_{q,p} = k_{p',q'}$
Krivine showed in 1977 that $k_{\infty,1} = \sqrt{2}$; his proof is not at all simple.

2. The starting point for investigating $k_{q,p}$ is the observation that

$$L_q^{\mathbb{C}}(\mu) = L_q^{\mathbb{R}}(\mu) \otimes_{\Delta_q} \mathbb{R}^2_2$$

holds isometrically where Δ_q is the norm induced on $L_q \otimes E$ by the space $L_q(E)$ of Bochner q-integrable functions (E a Banach space) – and the complexification of an operator $T \in \mathcal{L}(L_q^{\mathbb{R}}(\mu),\ L_p^{\mathbb{R}}(\nu))$ is just $T \otimes id_{\mathbb{R}^2_2}$. So general theorems of the continuity of

$$T \otimes S : L_q(\mu) \otimes_{\Delta_q} E \to L_p(\nu) \otimes_{\Delta_p} F$$

where $T \in \mathcal{L}(L_q(\mu), L_p(\nu))$ and $S \in \mathcal{L}(E, F)$ can be applied.

THEOREM 1 (Figiel–Iwaniec–Pelczyński):

$$k_{q,p} = 1 \quad \textit{whenever} \quad q \leq p.$$

3. This was the "good" case. For $q > p$ one can show that $k_{q,p}$ is the norm of $id_{\mathbb{R}^2_2}$ in the Banach operator ideal $\mathcal{L}_{p,q'}^{\text{inj sur}}$ – the injective and surjective hull of the ideal of (p, q')–factorable operators.

THEOREM 2: (1) *Let* $\infty \geq q \geq p \geq 1$.

$$(\sqrt{\pi})^{1/p - 1/q} \left(\frac{\Gamma\left(\frac{2+p}{2}\right)}{\Gamma\left(\frac{1+p}{2}\right)} \right)^{1/p} \cdot \frac{1}{2}\sqrt{\pi} \left(\frac{\Gamma\left(\frac{2+q'}{2}\right)}{\Gamma\left(\frac{1+q'}{2}\right)} \right)^{1/q'} \leq k_{q,p} \leq$$

$$\leq (\sqrt{\pi})^{1/p-1/q} \left(\frac{\Gamma\left(\frac{2+p}{2}\right)}{\Gamma\left(\frac{1+p}{2}\right)} \right)^{1/p} \left(\frac{\Gamma\left(\frac{2+q}{2}\right)}{\Gamma\left(\frac{1+q}{2}\right)} \right)^{-1/q}$$

(2) *In particular for* $1 \leq p \leq 2$

$$k_{2,p} = k_{p',2} = \frac{1}{\sqrt{2}} \left(\sqrt{\pi} \frac{\Gamma\left(\frac{2+p}{2}\right)}{\Gamma\left(\frac{1+p}{2}\right)} \right)^{1/p}$$

$$k_{2,1} = k_{\infty,2} = \frac{\pi}{\sqrt{8}} \approx 1,11072\ldots.$$

The proof of the lower estimate in (1) for example follows from $\mathrm{P}^{\mathrm{dual}}_{q'}(\mathbb{R}^2_2) \leq \mathrm{L}^{\mathrm{inj\,sur}}_{p,q'}(\mathbb{R}^2_2) \cdot \mathrm{I}_{p'}(\mathbb{R}^2_2)$ and trace duality.

Actually it turns out during the proof of this fact that it is enough to complexify operators

$$\mathbb{R}^n_q \to \mathbb{R}^n_p$$

– so the problem determining the exact value of the complexification constant $k_{q,p}$ (for $q > p$) is actually a *local* problem, in particular independent of the special measures.

PROPOSITION: *If* $q > p$ *then*

$$k_{q,p} = \alpha_{p,q'}(\cos(\cdot - \cdot\cdot);\ C([0,2\pi]), C([0,2\pi])$$

where $\alpha_{p,q'}$ is a certain tensor norm due to Lapresté. This formula was used by Krivine when he calculated $k_{\infty,1}$ (note that $\alpha_{1,1}$ is just the projective tensor norm π). It can be seen that $k_{\infty,1}$ is the real 2-dimensional Grothendieck constant $K^{\mathbb{R}}_G(2)$; the exact values of the higher dimensional Grothendieck constants are not yet known.

3. One central point for proving these results is using adequate isometric embeddings of $\mathbb{R}^2_2$ into some L_q. Using Lévy measures and ultraproducts the same methods which were used to prove theorem 1 (the "good" case) lead to

THEOREM 3: *Let* μ, ν *and* η *arbitrary measures and* $T \in \mathcal{L}(L_q(\mu), L_p(\nu))$. *Then*

$$\|T \otimes id_{L_r} : L_q(\mu) \otimes_{\Delta_q} L_r(\eta) \to L_p(\nu) \otimes_{\Delta_r} L_1(\eta)\| = \|T\|$$

whenever $q \leq p$ *and* r *satisfies one of the following five conditions:*
(1) $r = p$
(2) $r = 2$
(3) $q < r < 2$
(4) $r = q$
(5) $2 < r < p$.

For $q = p$ this is an old result of Marcinkiewicz-Zygmund.

The proofs of these results (as well as more references) the reader will find in [2] and [3].

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