Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

Boundedness of Fractional Integrals on Spaces of Homogeneous Type of Finite Measure

A. EDUARDO GATTO and STEPHEN VÁGI

I. Introduction

In this brief exposition we want to describe the main novelties in the extension of the theory of fractional integrals to spaces of homogeneous type with finite measure. The theory of fractional integrals was extended to R^n with Euclidean distance and a doubling weight in [GGW], and to homogeneous spaces of infinite measure in [GV1]. In [GV1] spaces of finite measure are considered, too, but the results apply only to function spaces with "homogeneous norms". This distinction between norms applies only to Lipschitz spaces and BMO, and due to duality, to H^p spaces with $\frac{1}{1+\gamma} < p \leq 1$ where γ is the order of the space. These spaces are, in fact, the ones where the novelties alluded to above occur, and we now list them. A condition based on the smoothness of the function $I_\alpha 1$ where I_α is the fractional integral of order α turns out to be necessary and sufficient for boundedness of I_α on Lipschitz spaces and H^p spaces, $p < 1$. The corresponding result for H^1 is related to the boundedness of I_α from BMO to $Lip[\alpha]$. In these two cases a sufficient condition is given which again is based on the smoothness of $I_\alpha 1$. Finally, for H^p spaces the image under I_α of a general p-atom with or without vanishing integral may not have integral zero, but it can be shown that it is a molecule as defined in section II below. In section II we state the results in detail, and in section III we sketch the proofs. Complete proofs will appear in [GV2].

II. Statement of Results

In the sequel (X, δ, μ) is a normal space of homogeneous type of finite measure and of order γ, $0 < \gamma \leq 1$. We may, and will assume that $\mu(X) = 1$. The diameter of the space is finite and will be denoted by D. $B_r(x)$ will denote the ball of center x and radius r, i.e., $B_r(x) = \{y \epsilon X : \delta(x, y) < r\}$. In order to define the kernel of the fractional integral without having to distinguish the case when the measure μ has atoms we shall adopt the following abuse of notation. If $0 < \alpha < 1$, we define

$$\frac{1}{\delta(x,y)^{1-\alpha}} = \begin{cases} \frac{1}{\delta(x,y)^{1-\alpha}} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The fractional integral of order $\alpha, 0 < \alpha < 1$, on measurable functions f is defined by

$$I_\alpha f(x) = \int_X \frac{f(y)d\mu(y)}{\delta(x,y)^{1-\alpha}}$$

We shall now give the definition of molecules:

Let $\epsilon > 0$ and $(1+\gamma)^{-1} < s < 1$. A measurable function M on X is an s-molecule if there exist a point x_o in X and constants $L > 0$ and $r > 0$ such that

$$| M(x) | \leq L\ \mu(B_r(x_o))^{-1/s} \quad \text{for all} \quad x, \tag{2.1}$$

$$| M(x) |\ \delta(x_1, x_o)^{\frac{1}{s}+\epsilon} \leq L\ \mu(B_r(x_o))^{\epsilon} \tag{2.2}$$

and

$$| \int_X M d\mu | \leq L. \tag{2.3}$$

For s = 1, (2.1) and (2.2) are the same but (2.3) is replaced by

$$| \int_X M d\mu | \leq \frac{L}{1+ | \log \mu(B_r(x_o)) |}. \tag{2.4}$$

The non-homogenous Lipschitz norms and BMO norm mentioned in the introduction are

$$\|f\|_{Lip[\beta]} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\delta(x,y)^\beta}$$

$$\|f\|^* = \|f\|_1 + \sup_B m_B(|f - m_B(f)|)$$

where $m_B(h) = \frac{1}{\mu(B)} \int_B h(x)d\mu(x)$, and B ranges over all balls in X. The letter c will denote a constant, not neccesarily the same at each occurence.

We now state the theorems:

Theorem 1. Let $(1+\gamma)^{-1} < s \leq 1$ and let M be an s-molecule. Then M belongs to H^s and

$$\|M\|_{H^s} \leq C.$$

Where C is a constant that depends only on s, ϵ and L for $s < 1$, and on ϵ and L for $s = 1$.

Theorem 2. Let $0 < \alpha+\beta < \gamma$, and $f \in Lip[\beta]$ then the following statements are equivalent:
i) $I_\alpha 1 \in Lip[\alpha + \beta]$.

ii) $I_\alpha f(x)$ converges absolutely for every x and there is a constant c independent of f such that

$$\|I_\alpha f\|_{Lip[\alpha+\beta]} \leq c\|f\|_{Lip[\beta]}.$$

Theorem 3. Let $0 < \alpha < \gamma$ and $f \in$ BMO. If $I_\alpha 1$ satisfies

$$\sup_{x \neq y} \frac{| I_\alpha 1(x) - I_\alpha 1(y) | (1 + | \log \delta(x,y) |)}{\delta(x,y)^\alpha} = c_{I_\alpha 1} < \infty \tag{2.5}$$

then $I_\alpha f(x)$ converges absolutely for every x in X, and there exists a constant c independent of f such that

$$\|I_\alpha f\|_{Lip[\alpha]} \leq c \|f\|^*$$

Theorem 4. Let $(1+\gamma)^{-1} < p < 1$, $0 < \alpha < \gamma$ and $1 < \frac{1}{q} = \frac{1}{p} - \alpha$. Then the following statements are equivalent:
i) $I_\alpha 1 \in Lip[\frac{1}{p} - 1]$
ii) For any p-atom a, $I_\alpha a$ belongs to H^q, and there exists a constant c independent of a such that

$$\|I_\alpha a\|_{H^q} \leq c,$$

and I_α extends to a continuous linear map from H^p to H^q.

Theorem 5. Let $0 < \alpha < \gamma$. If $I_\alpha 1$ satisfies (2.5) then for any $(1+\alpha)^{-1}$ – atom a, $I_\alpha a$ belongs to H^1, there is a constant c independent of a such that

$$\|I_\alpha a\|_{H^1} \leq c,$$

and I_α extends to a continous linear map from $H^{1/(1+\alpha)}$ to H^1.

III. Sketch of the Proofs.

Theorem 1. We begin by decomposing the molecule M as the following sum

$$M(x) = M_o(x) + J\frac{1}{\mu(B)}\chi_B(x),$$

where $J = \int M d\mu$ and χ_B is the characteristic function of $B = B_r(x_o)$. The function M_o is a molecule with vanishing integral and therefore by the results of Coifman and Weiss [GW] it is in H^s, and $\|M_o\|_{H^s} \leq C_1$. The second term is a particular molecule with non-vanishing integral. Its H^s norm can be estimated using the β-maximal function of Macias and Segovia [MS2] or, as suggested by Professor Taibleson at this conference, by decomposing it into a finite sum of atoms in the following way. Let $B_k = B_{2^k r}(x_o), k = 0, 1, ...K$, where $B_K = X$ but $B_{K-1} \neq X$. It is easy to see that $K \leq c_1(1 + |\log \mu(B)|)$. For $k = 1, 2, ...K$, let

$$b_k = \frac{1}{\mu(B_{k-1})}\chi_{B_{k-1}} - \frac{1}{\mu(B_k)}\chi_{B_k}.$$

Then $b_k = \lambda_k a_k$, where $\lambda_k = \mu(B_k)^{1/s} / \mu(B_{k-1})$ and a_k is an s-atom. One checks that

$$J\frac{\chi_B}{\mu(B)} = \sum_{k=1}^{K} J\lambda_k a_k + J.$$

From this decomposition it follows that

$$\|J\frac{1}{\mu(B)}\chi_B\|_{H^s} \leq C_2.$$

The constants C_1 and C_2 depend only on L ϵ, and s.

Theorem 2. The implication ii) $\Rightarrow$ i) is immediate from the fact that $1 \epsilon Lip[\beta]$. To prove that i) $\Rightarrow$ ii) we write for $x_1 \neq x_2$

$$I_\alpha f(x_1) - I_\alpha f(x_2) = \int_X [\frac{1}{\delta(x_1, y)^{1-\alpha}} - \frac{1}{\delta(x_2, y)^{1-\alpha}}](f(y) - f(x_1))d\mu(y)$$

$$+f(x_1)[I_\alpha 1(x_1) - I_\alpha 1(x_2)].$$

The first term is estimated by a standard argument and we obtain that its absolute value is less than or equal to $c \ \|f\|_{Lip[\beta]} \ \delta(x_1, x_2)^{\alpha+\beta}$. Using i) we show that the second term in absolute value is less than or equal to $c\|f\|_\infty \ \delta(x_1, x_2)^{\alpha+\beta}$. On the other hand $\|I_\alpha f\|_\infty \leq c\|f\|_\infty$ because the kernel of I_α is in L^1.

Theorem 3. For $x_1 \neq x_2$, let $r = \delta(x_1, x_2), B = B_r(x_1)$, and write

$$I_\alpha f(x_1) - I_\alpha f(x_2) =$$

$$\int_X [\frac{1}{\delta(x_1, y)^{1-\alpha}} - \frac{1}{\delta(x_2 y)^{1-\alpha}}][f(y) - m_B(f)]d\mu(y)$$

$$+ m_B(f)[I_\alpha 1(x_1) - I_\alpha 1(x_2)].$$

The first term is estimated by a standard argument, and we obtain that its absolute value is less than or equal to $c\delta(x_1, x_2)^\alpha \|f\|^*$. The estimate of the second term depends on the inequality

$$|m_B(f)| \leq c\|f\|^*(1 + |\log \mu(B)|)$$

valid for f in BMO. This inequality together with (2.5) imply that the second term in absolute value is less than or equal to $c\|f\|^*\delta(x_1,x_2)^\alpha$.

Theorem 4. The implication ii) $\Rightarrow$ i) follows easily from Theorem 2 and duality. To prove that i) $\Rightarrow$ ii) we first consider a p-atom a whose support is contained in ball centered at x_o, with $\int a d\mu = 0$. The proofs of (2.1) and (2.2) are done by standard arguments. To prove (2.3) we use $\int a d\mu = 0$ and a change of order of integration to write

$$\int I_\alpha a(x) d\mu(x) = \int [\int \frac{a(y)}{\delta(x_1 y)^{1-\alpha}}\, d\mu(y)] d\mu(x) =$$

$$\int a(y) I_\alpha 1(y)\, d\mu \;=\; \int a(y)[I_\alpha 1(y) - I_\alpha 1(x_0)] d\mu(y).$$

Since $I_\alpha 1$ belongs to $Lip[\frac{1}{p}-1]$, and α is a p-atom in supported $B_r(x_o)$ we have

$$|\int I_\alpha a d\mu| \;\leq\; \|I_\alpha 1\|_{Lip[\frac{1}{p}-1]} r^{\frac{1}{p}-1} \int |a| d\mu \;\leq\; D^{1/p-1}\|I_\alpha 1\|_{Lip[\frac{1}{p}-1]} = L.$$

It is easy to see that $I_\alpha 1$ is an H^q molecule, too. It is then shown by standard arguments involving the atomic decomposition that I_α entends to all H^p as a bounded map into H^q.

Theorem 5. Again let a be a $(1+\alpha)^{-1}$ - atom supported in a ball B of center x_0. The proof of (2.1) and (2.2) are done by standard arguments. To prove (2.4) we use $\int \alpha d\mu = 0$ and a change of order of integration to write, as in the proof of Theorem 4

$$\int I_\alpha a(x) d\mu(x) = \int a(y)[I_\alpha 1(y) - I_\alpha 1(x_0)] d\mu(y).$$

Now we use (2.5) to obtain

$$|\int I_\alpha a d\mu| \;\leq\; c\, \frac{1}{1+|\log(\mu(B))|}.$$

It is easy to see that $I_\alpha 1$ is a 1 - molecule. Finally the extension of I_α to all of $H^{\frac{1}{1+\alpha}}$ is proved as in Theorem 4.

References

[CW] Coifman, R.R. and Weiss, G., Extensions of Hardy spaces and their use in analysis, Bull. of the Am. Math. Society, 83(1977), pp. 569-645.

[GGW] Gatto, A.E., Gutiérrez, C., Wheeden, R., On Weighted Fractionals Integrals, in Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol 1, pp. 124-137, ed. by W. Beckner et al.) Wadsworth, Inc., Belmont, CA 1983.

[GV1] Gatto, A.E. and Vági, S., Fractional Integrals on Spaces of Homogeneous type, in Analysis and Partial Differential Equations, pp.171-216, ed. by Cora Sadosky, Marcel Dekker, New York and Basel, 1990.

[GV2] Gatto, A.E. and Vági, S., On Molecules and Fractional Integrals on spaces of homogeneous type with finite measure. To appear in Studia Mathematica.

[MS1] Macías, R.A., and Segovia, C., Singular integrals on generalized Lipschitz and Hardy spaces, Studia Math., 65(1979), pp. 257-270.

[MS2] Macías, R.A., and Segovia, C., A decomposition into atoms of distributions on spaces of homogeneous type, Advances in Math., 33(1979), pp. 271-309.

[TW] Taibleson, M.H., and Weiss, G., The molecular characterization of Hardy spaces, Astérisque, 77(1980), pp. 66-149.

DePaul University