Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

NONRESONANCE IN SOME SEMILINEAR NEUMANN PROBLEMS

JEAN-PIERRE GOSSEZ

1. Let Ω be a bounded open subset in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Let f be a continuous function from $\mathbf{R}$ to $\mathbf{R}$. We consider the semilinear Neumann problem

$$(1) \qquad \begin{cases} -\Delta u = f(u) + h(x) \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

where ν denotes the unit exterior normal. We are interested in the conditions to be imposed on the nonlinearity f in order that problem (1) admits at least one solution $u(x)$ for any given $h(x)$. Such conditions are usually called nonresonance conditions.

In the linear case $f(u) = \lambda u$, the equation in (1) reads $-\Delta u = \lambda u + h(x)$. A necessary and sufficient condition for (1) to be solvable for any $h(x)$ (say in $L^2(\Omega)$) is then clearly that λ be different from the (distinct) eigenvalues $\lambda_1 = 0 < \lambda_2 < \lambda_3 < \ldots$ of $-\Delta$ with homogenous Neumann boundary conditions.

In the nonlinear case, the equation in (1) formally reads $-\Delta u = (f(u)/u)u + h(x)$ and we see that it is the quotient $f(u)/u$ which plays the role of the above number λ. One can then guess that if this quotient does not interfere too much with the spectrum $\lambda_1 < \lambda_2 < \ldots$, then nonresonance should occur.

It is our purpose in this talk to review some recent results in that direction and to mention some related open questions.

2. Our starting point will be the classical result of Dolph [Do]: if, for some k,

$$(2) \qquad \lambda_k < \liminf_{s\to\pm\infty} \frac{f(s)}{s} \leq \limsup_{s\to\pm\infty} \frac{f(s)}{s} < \lambda_{k+1},$$

then (1) is solvable for any h in $L^2(\Omega)$. The proof goes by degree theory. In fact (2) implies an a priori bound in $H^1(\Omega)$ for all solutions of (1).

Condition (2) of Dolph was weakened by Costa-Oliveira into a condition which involves the primitive of the nonlinearity: $F(s) = \int_0^s f$. Solvability of (1) for any h in $L^2(\Omega)$ is derived in [Co-Ol] under the following two assumptions:

$$(3) \qquad \lambda_k \leq \liminf_{s\to\pm\infty} \frac{f(s)}{s} \leq \limsup_{s\to\pm\infty} \frac{f(s)}{s} \leq \lambda_{k+1},$$

$$\lambda_k < \liminf_{s\to\pm\infty} \frac{2F(s)}{s^2} \le \limsup_{s\to\pm\infty} \frac{2F(s)}{s^2} < \lambda_{k+1}. \tag{4}$$

The proof here is variational. Observe that there is no a priori bound in general (think of $k = 0$, $h = 0$, with f having an unbounded set of zeros, which yield constant solutions of (1)). Condition (3) and (4) are used to prove that the corresponding functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u) - \int_\Omega hu, \qquad u \in H^1(\Omega)$$

satisfies the (P.S.) condition and has the right shape in order to apply the Rabinowitz saddle point theorem.

The case where equality holds in the extreme left or right inequalities of (4) was also considered recently. Various sufficient conditions for nonresonance in such cases can be found in [Co], [Si], [Ra], [Mo].

3. We wish now to describe a result near the first eigenvalue $\lambda_1 = 0$ which provides a necessary and sufficient condition for nonresonance. Let us assume that no interference occurs with the higher part of the spectrum, in the following sense (inspired from (3), (4)):

$$\limsup_{s\to\pm\infty} \frac{f(s)}{s} \le \lambda_2, \tag{5}$$

$$\limsup_{s\to\pm\infty} \frac{2F(s)}{s^2} < \lambda_2. \tag{6}$$

Then a necessary and sufficient condition for (1) to be solvable for any $h \in L^\infty(\Omega)$ is that $f : \mathbf{R} \to \mathbf{R}$ be unbounded from above and from below (cf. [Go-Om$_2$]).

Necessity follows immediately by integrating the equation in (1). Sufficiency is proved by using degree theory, by constructing a bounded open set $\mathcal{O}$ in $C^1(\bar{\Omega})$ which contains o and which is such that no solution of the family of homotopic problems

$$\begin{cases} -\Delta u = (1-\mu)\theta u + \mu[f(u) + h(x)] \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

arises on the boundary $\partial\mathcal{O}$. (Here θ is fixed with $\lambda_1 = 0 < \theta < \lambda_2$ and μ varies with $0 \le \mu < 1$). One should observe here that no a priori bound holds in general. Actually even the (P.S.) condition fails in general (think of $h = 0$, with F uniformly bounded and f having an unbounded set of zeros, which provide an unbounded (P.S.) sequence). In this respect obtaining a variational proof of the above result would certainly be of interest.

Condition (6) on the primitive F of the nonlinearity f is used through its equivalence with a positive density condition. Precisely, assuming (5) and that f has at most linear growth, then (6) holds if and only if there exists $\eta > 0$ such that the set

$$E = \left\{ s \in \mathbf{R}_0;\ \frac{f(s)}{s} \le \lambda_2 - \eta \right\}$$

has a positive density at $+\infty$ and at $-\infty$. This means that

$$\liminf_{r\to+\infty} \frac{|E\cap[0,r]|}{|[0,r]|} > 0, \liminf_{r\to-\infty} \frac{|E\cap[r,0]|}{|[r,o]|} > 0,$$

where $|\ |$ denotes Lebesgue measure. This notion of positive density was introduced in [DF-Go] and the above equivalence was proved in [Go-Om$_1$].

4. The results of [Do], [Co-Ol], [Co], [Si], [Ra], [Mo] mentionned above also hold for the Dirichlet problem (and actually were originally derived in that setting). This is not so however for the result of [Go-Om$_2$]. Consider the problem

$$(7) \qquad \begin{cases} -\Delta u = \lambda_1 u + f(u) + h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $0 < \lambda_1 < \lambda_2 < \ldots$ now denotes the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$. Multiplying by the first (positive) eigenfunction and integrating, one immediately sees that a necessary condition for nonresonance is again that $f : \mathbf{R} \to \mathbf{R}$ be unbounded from above and from below. This condition however is no more sufficient (assuming of course no interference with the higher part of the spectrum), even in the ODE case, as was shown recently by Njoku [Nj]. No necessary and sufficient condition seems to be known in the case of the Dirichlet problem.

5. The proof of the result discussed in §3 involves the consideration of constant lower and upper solutions which may not be well ordered. We will now state a general result in that direction.

Let us consider the problem

$$(8) \qquad \begin{cases} -\Delta u = f(x,u) \text{ in } \Omega, \\ \text{Dirichlet or Neumann homogeneous condition on } \partial\Omega, \end{cases}$$

where f is a L^p Caratheodory function for some $p > N$. A classical result says that if (8) admits a lower solution $\alpha(x)$ and an upper solution $\beta(x)$, with $\alpha(x) \le \beta(x)$ in Ω, then (8) admits a solution $u(x)$, with $\alpha(x) \le u(x) \le \beta(x)$. See e.g. [Am]. Assume now that

$$(9) \qquad \lambda_1 \le \liminf_{s\to\pm\infty} \frac{f(x,s)}{s} \le \limsup_{s\to\pm\infty} \frac{f(x,s)}{s} \lneqq \lambda_2,$$

where $\lambda_1 < \lambda_2 < \ldots$ denote the eigenvalues of $-\Delta$ under the corresponding boundary conditions. Then it is shown in [Go-Om$_3$] that the sole existence of a lower solution $\alpha(x)$ and of an upper solution $\beta(x)$ (with possibly no ordering relation between them) implies the existence of a solution $u(x)$.

This allows to recover in a different way the characterization of nonresonance discussed in §3, under however a noninterference condition with respect to λ_2 which is a little bit stronger than (5), (6). Weakening the right inequality in (9) so as to reach a condition of the type (5), (6) remains unclear at this moment. Suppressing the restriction with respect to λ_1 in (9) also remains unclear at this moment (this is possible in the case of the Neumann problem when $f(x,u)$ splits as $f(u) + h(x)$ with $h \in L^\infty(\Omega)$).

Another question concerns the possibility of considering in §3 forcing terms in $L^2(\Omega)$ instead of $L^\infty(\Omega)$. The technical difficulty is related to the use of constant lower and upper solutions.

REFERENCES

[Am] H.AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces,* Siam Review, 18 (1976), 620-709.

[Co] D.COSTA, *A note on unbounded perturbations of linear resonant problems,* to appear.

[Co-Ol] D.COSTA and A.OLIVEIRA, *Existence of solution for a class of semilinear elliptic problems at double resonance,* Boll. Soc. Bras. Mat., 19 (1988), 21-37.

[DF-Go] D.DE FIGUEIREDO and J.-P.GOSSEZ, *Conditions de non résonance pour certains problèmes elliptiques semi linéaires,* C.R. Ac. Sc. Paris, 302 (1986), 543-545.

[Do] C.DOLPH, *Nonlinear integral equations of Hammerstein type,* Trans. Am. Math. Soc., 66 (1949), 289-307.

[Go-Om$_1$] J.-P.GOSSEZ and P.OMARI, *Periodic solutions of a second order ordinary differential equation: a necessary and sufficient condition for nonresonance,* J. Diff. Equat., to appear.

[Go-Om$_2$] J.-P.GOSSEZ and P.OMARI, *A necessary and sufficient condition of nonresonance for a semilinear Neumann problem,* Proc. Am. Math. Soc., to appear.

[Go-Om$_3$] J.-P.GOSSEZ and P.OMARI, *Nonordered lower and upper solutions in semilinear elliptic problems,* to appear.

[Mo] M.MOUSSAOUI, *Questions d'existence dans les problèmes semi-linéaires elliptiques,* Thèse Doct., Université Libre de Bruxelles, 1991.

[Ra] M.RAMOS, *Remarks on resonance problems with unbounded perturbations,* to appear.

[Si] E.A.SILVA, *Critical point theorems and applications to differential equations,* Ph. D. thesis, Univ. Wisconsin, 1988.

Département de Mathématique
Campus Plaine C.P.214
Université Libre de Bruxelles
1050 Bruxelles - Belgique