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LOCAL SOLVABILITY OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Where it all begins

The most primitive question one can ask concerning a partial differential equation is if there exists a solution, at least locally, and not subjected to any additional condition. For ordinary differential equations, very satisfactory theorems stating local existence of solutions under very mild hypothesis of regularity had been known since long ago, and it came as a surprise when Hans Lewy discovered, in 1956, his now famous example of a first-order linear equation whose coefficients are polynomial of degree at most one, failing to have local solutions. Indeed, if $f \in C^{\infty}(\mathbb{R}^3)$ is conveniently chosen, the equation

$$(\partial_x + i\partial_y - (x+iy)\partial_z)u = f \qquad (x,y,z) \in \mathbb{R}^3$$

does not have distribution solutions in any open subset of $\mathbb{R}^3$ ([L]).

For elliptic equations local solvability was known and Hörmander had proved in is thesis that linear operators of real principal type were locally solvable. Let us recall some definitions and notation. A linear partial differential operator in an open subset Ω of $\mathbb{R}^n$ has the form

$$P(x,D)u = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u, \qquad u \in C_c^\infty(\Omega), \tag{1.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ denotes a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ its length, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $D_j = \partial/i\partial x_j$. The principal symbol of $P(x,D)$ is the function

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \qquad x \in \Omega, \qquad \xi \in \mathbb{R}^n.$$

Here, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. If we interpret $p_m(x,\xi)$ as defined on the cotangent bundle $T^*(\Omega)$ rather than on the modest cartesian product $\Omega \times \mathbb{R}^n$, the principal symbol becomes invariantly defined under change of coordinates. A linear partial differential operator is elliptic if its principal part has no real zeros $\xi \neq 0$ (of course, $p_m(x,0) = 0$, $x \in \Omega$, because it is a homogeneous polynomial in ξ, but these zeros, being trivial, do not count). The main difficulty in finding a right inverse for a linear PDE comes from the real nontrivial zeros of the principal part. In the elliptic case those zeros simply not occur. The best thing next to

* *Partly supported by CNPq*

no zeros is simple zeros. An operator is said to be of principal type if

$$p_m(x,\xi)=0, \quad \xi\in\mathbb{R}^n\setminus 0 \quad \text{implies} \quad \nabla p_m(x,\xi)\neq 0.$$

At this point it is perhaps convenient to introduce a formal definition.

Definition 1. *A partial differential operator $P(x,D)$ in $\Omega\subset\mathbb{R}^n$ is locally solvable if every point $x_0\in\Omega$ has a neighborhood $U\subset\Omega$ such that the equation*

$$P(x,D)u=f \tag{1.2}$$

can be solved in $\mathcal{D}'(U)$ for every $f\in C_c^\infty(U)$.

If the coefficients of p_m are real then the operator is of real principal type and by the theorems of Hörmander [H1] it is locally solvable. Notice that Lewy's operator $\partial_x+i\partial_y-(x+iy)\partial_z$ is of principal type but does not have real coefficients. A few years later Hörmander ([H2], [H3]) generalized Lewy's example in the following way. Denote by $\overline{P}(x,D)$ the operator obtained from $P(x,D)$ by replacing each coefficient by its complex conjugate and consider the commutator

$$C(x,D)=[P(x,D),\overline{P}(x,D)]=P(x,D)\overline{P}(x,D)-\overline{P}(x,D)P(x,D)$$

which is an operator of order $2m-1$ with principal symbol $c_{2m-1}(x,\xi)$. If $P(x,D)$ is locally solvable in Ω then $c_{2m-1}(x,\xi)$ must vanish at every zero of $p_m(x,\xi)$ in $\Omega\times\mathbb{R}^n$. An operator satisfying the latter condition will be said to satisfy condition $(\mathcal{H})$. For the Lewy operator condition $(\mathcal{H})$ is violated at every point. If the coefficients of $P(x,D)$ are real or constant $c_{2m-1}(x,\xi)$ vanishes identically. This was a most remarkable advance because explained what appeared as an isolated example in terms of very general geometric properties of the symbol (an invarianlty defined object). However, it was not accurate enough to discriminate solvable from nonsolvable among some examples considered by Mizohata [M], which we now describe. Let k be a positive integer and consider the operator in $\mathbb{R}^2$ defined by

$$M_k=\frac{\partial}{\partial y}-iy^k\frac{\partial}{\partial x}.$$

If $k=1$ condition $(\mathcal{H})$ is violated at all points of the x axis so, in particular, M_1 is not locally solvable at the origin. For $k\geq 2$ condition $(\mathcal{H})$ is satisfied everywhere. On the other hand, it turns out by relatively simple arguments that M_k is locally solvable at the origin if and only if k is even ([Gr], [Ga]). The principal symbol of M_k is $m_1=-i(\eta-iy^k\xi)$. The crucial difference between k odd and k even is that in the first case the function y^k changes sign and in the second case doesn't. Nirenberg and Treves [NT1] elaborated over these examples and came up with which turned out to be the right condition for local solvability of operators of principal type. Write the principal symbol in terms of real and imaginary parts, $p_m(x,\xi)=a(x,\xi)+ib(x,\xi)$. A null bicharacteristic of $a(x,\xi)$ is a curve satisfying the system of ordinary differential equations

$$\begin{cases} \dot{x} = \nabla_\xi a(x,\xi), & x(0) = x_0, \\ \dot{\xi} = -\nabla_x a(x,\xi), & \xi(0) = \xi_0, \end{cases}$$

with initial conditions verifying $a(x_0,\xi_0) = 0$. The operator (1.1) satisfies condition $(\mathcal{P})$ if $b(x,\xi)$ does not change sign along any null bicharacteristic of $a(x,\xi)$ (if the projection of the bicharacteristic into Ω is singular one has to interchange the roles of a and b which amounts to multiplying $P(x,D)$ by i). This formulation is invariant by coordinate changes. When condition $(\mathcal{H})$ is violated the change of sign occurs at a simple zero, but $(\mathcal{P})$ detects changes of sign at zeros of higher order, as in the Mizohata operators M_k for $k > 1$ odd, including infinite order. For first-order operators of principal type with real analytic coefficients in the principal part they proved that $(\mathcal{P})$ is equivalent to local solvability. That was in 1963 and seven years later they were able to extend the same result to operators of arbitrary order m [NT2]. The proof of the sufficiency involved a microlocalization used to split the operator as a finite sum of simpler operators. Each simpler operator could be further reduced by factoring out an elliptic operator of order $m-1$. The remaining factor had order one and still verified condition $(\mathcal{P})$. It should be said, however, that the new operators introduced by microlocalization were no longer differential but pseudo-differential operators. Each first-order pseudo-differential factor was again simplified by conjugation with a Fourier integral operator. Property $(\mathcal{P})$ can be used at this stage to obtain *a priori* estimates. They are sufficiently strong to survive the microlocalization and can be patched together to obtain a local estimate for the original operators. This implies that the transpose of $P(x,D)$ is locally solvable (interchanging the roles of $P(x,D)$ and its transpose one proves that $P(x,D)$ itself is locally solvable) Three years later R. Beals and C. Fefferman [BF] proved that smooth coefficients were enough to show that $(\mathcal{P})$ implies local solvability. In fact, they started from the reduced operators of Nirenberg and Treves and used a finer technique of pseudo-differential operators that included the Calderón-Vaillancourt result that a pseudo-differential operator with symbol in the class $S^0_{1/2,1/2}$ is bounded in L^2 [CV].

Concerning the necessity of $(\mathcal{P})$ Moyer [Mo] removed in 1978 the analiticity hypothesis for operators in two variables and his technique was applied by Hörmander [H4] to extend the result for operators in any number of variable with smooth coefficents. Finally, the conjecture of Nirenberg and Treves that $(\mathcal{P})$ was equivalent to local solvability for operators with smooth coefficients had been proved over a span of 15 years. The technique of Nirenberg-Treves and Beals-Feffereman actually gave more than solvability in the class of distributions: if one takes $f \in H^s_c(U)$, $s > -n/2$ in (1.2) then one can choose u in H^{s+m-1}. This is not enough to furnish smooth solutions when $f \in C^\infty_c(U)$ because the diameter of U shrinks as s grows in the proofs. On the other hand Hörmander [H5] gave a different proof by studying the propagation of singularities of operators that verify $(\mathcal{P})$, which allowed to obtain semi-global solutions, i.e., solutions defined on a full compact set (under the geometric assumption that bicharacteristics do not get trapped in the given compact set). Furthermore, the solutions can be taken smooth if f is smooth.

2. Nonlinear equations

A fully nonlinear operator of order m acting on functions u is of the form

$$u(x) \mapsto F(x, \partial^\alpha u(x)), \qquad |\alpha| \le m. \tag{2.1}$$

Two different situations arise. If $F(x,\zeta)$ is a function defined in $\Omega \subset \mathbb{R}^n \times \mathbb{R}^N$ (where N is the number of α's $\in \mathbf{Z}^n_+$ such that $|\alpha| \le m$) only real functions u are allowed in (2.1). On the other hand, if $\Omega \subset \mathbb{R}^n \times \mathbb{C}^N$ and $F(x,\zeta)$ is a holomorphic function of ζ, the natural functions $u(x)$ in (2.1) are complex valued. At any rate, the linearization of a nonlinear operator $F(u)$ at a function u is the linear differential operator $F'(u)$ defined by

$$F'(u)v = \frac{d}{dt}F(u+tv)\Big|_{t=0}.$$

In general, an equation such as $F(x, \partial^\alpha u(x)) = f(x)$ may fail to have a solution by very simple reasons: if the set of values of F and the set of values of f are disjoint (for instance, if $F > 0$ and $f \le 0$) there can be no solution. So a better problem is the following: given u_0 and setting $f_0(x) = F(x, \partial^\alpha u_0(x))$, try solving

$$F(x, \partial^\alpha u(x)) = f(x) \tag{2.2}$$

for f sufficiently close to f_0 in some topology, if possible, with u close to u_0. Now this is an implicit function problem and a very powerful tool to handle it is the Nash-Moser implicit function theorem [Na], [Mos], [Ha]. In order to apply it one has to construct an operator $Q(u)v$ acting on some tame scale of spaces (typically, the scale of Sobolev spaces H^s) such that

$$F'(u)Q(u)v = v, \qquad v \in H^s, \quad s > s_0 \tag{2.3}$$

$$\|Q(u)v\|_s \le C_s(\|u\|_{s+r}\|v\|_{s_0} + \|u\|_{s_0}\|v\|_{s+r}), \qquad u, v \in H^{s+r} \tag{2.4}$$

There are other hypotheses in the Nash-Moser theorem but they are almost automatic when dealing with differential operators like (2.1). Note that (2.3) states that the linear differential operator $F'(u)$ has a right inverse, so if $F'(u)$ is locally solvable for each u we might be able to construct some right inverse $Q(u)$. It is with condition (2.4) that the trouble really comes: the coefficients of the linear operator $F'(u)$ depend on u and the norm of $Q(u)$ must grow in a specific way (tamely). In order to obtain tame estimates one usually needs explicit expressions, involving pseudo-differential and Fourier integral opertors (the usual techniques to get *a priori* energy estimates do not seem useful when it comes to obtaining tame estimates). Fortunately, there exists a tame calculus for pseudo-differential and Fourier integral opertors that makes of them an appropriate tool for the task when they can be used ([GY], [AH]). In the important case in which F is smooth in all arguments and $F'(u)$ is of real principal type for every u, this approach allows to solve (2.2) locally, with u smooth if f is smooth [GY]. This has interesting applications, particularly to nonlinear equations arising Riemannian geometry.

For the case of complex equations much less is known. There are results only for two variables [D], [AH] and the assumptions on $F'(u)$ are considerably more stringent than simply assuming that it is locally solvable. For instance, if $b(x,t)$ is a smooth real function in $\mathbb{R}^2$ such that $t \to b(x,t)$ does not change sign, then $L = \partial_t + ib(x,t)\partial_x$ is linear and verifies $(\mathcal{P})$ but it does not seem to be known if there exists a right inverse Q for L such that $\|Qf\|_s \leq C_s\|f\|_{s+r}$ for any positive s and f smooth and supported in a *fixed* neighborhood unless $b(x,t)$ does not change sign at all ([AH]). In the general case, the parametrices constructed by the methods of Treves [T1], [T2] or even those which are valid in a more restricted set-up [AH], [Ho2], seem to verify only the weaker estimates $\|Qf\|_s \leq C_s\|f\|_{2s+r}$ which are not enough to guarantee the convergence of the Nash-Moser scheme.

3. Nondetermined systems of vector fields

The solvability theorems valid for one equation of principal type can be extended to determined systems (principal type for a system means that the determinant of the matrix principal part has at most simple (nontrivial) real zeros). For nondetermined systems the theory is at a primitive stage of developement and only special equations involving vector fields and related to the familiar gradient and divergence equations can be dealt with in some generality. Important classes of such equations arise naturally in the theory of holomorphic functions of several variables ($\overline{\partial}$ and $\overline{\partial}_b$ equations).

Let L_j, $j = 1, \ldots, n$ be linearly independent complex vector fields defined on the open set $\Omega \subset \mathbb{R}^N$, i.e.,

$$L_j = \sum_{k=1}^{N} a_{jk}(x)\frac{\partial}{\partial x_k}, \qquad a_{jk} \in C^\infty(\Omega).$$

The basic question is to determine when overdetermined or underdetermined equations like

$$\begin{cases} L_1 u = f_1, \\ L_2 u = f_2, \\ \ldots\ldots\ldots \\ L_n u = f_n, \end{cases} \qquad f_1, \ldots, f_n \in C^\infty(\Omega), \tag{3.1}$$

or

$$L_1 u_1 + \cdots + L_n u_n = f, \qquad f \in C^\infty(\Omega), \tag{3.2}$$

can be solved for any choice of the right hand side satisfying the proper compatibility conditions. For instance, if the vectors L_j commute pairwise, the compatibility conditions will be $L_j f_k = L_k f_j$ for the system (3.1) and void for (3.2). In the case of a single vector field, ($n = 1$), (3.1) and (3.2) coincide and the answer depends on condition $(\mathcal{P})$ of Nirenberg and Treves ([NT1]). Since it is possible to obtain equivalent equations by replacing each L_j by a linear combination of the vectors L_j, so that corresponding matrix is non-singular, we realize that the relevant geometric object will be the vector bundle $\mathcal{L} \subset \mathbb{C} \otimes T\Omega$ generated by $L_1, \ldots, L_n$. We say that a sub-bundle of $\mathcal{L} \subset \mathbb{C} \otimes T\Omega$ is involutive (or formally integrable) if it satisfies the Frobenius condition

$$[\mathcal{L}\,,\mathcal{L}\,] \subseteq \mathcal{L}\,. \tag{3.3}$$

In this case, if $\mathcal{L}^{\perp}$ denotes the orthogonal of $\mathcal{L}$ relative to the duality between tangent vectors and differential forms of degree 1 ($\mathcal{L}^{\perp}$ is a sub-bundle of the complexified cotangent bundle), the exterior derivative defines, by passage to the quotient, a complex of diferential operators

$$C^{\infty}(\Omega, \Lambda^{p}(\mathbb{C} \otimes T^{*}\Omega/\mathcal{L}^{\perp})) \xrightarrow{\delta_p} C^{\infty}(\Omega, \Lambda^{p+1}(\mathbb{C} \otimes T^{*}\Omega/\mathcal{L}^{\perp})), \qquad p = 0, 1, \ldots, n-1. \tag{3.4}$$

In local coordinates, equations $\delta_0 u = f$ and $\delta_{n-1} u = f$ can be expressed in the form (3.1) and (3.2) respectively.

The cohomology groups of sequence (3.4)

$$\mathcal{H}^{q}(\Omega, \mathcal{L}\,), \qquad q = 0, 1, \ldots, n$$

are called the cohomology grups associated to $\mathcal{L}$ on Ω and a natural problem of the theory is to determine when they vanish for a given structure $\mathcal{L}$. The localized problem is also relevant. If we fix a point $A \in \Omega$ we may consider the associated complex

$$C^{\infty}(A, \Lambda^{p}(\mathbb{C} \otimes T^{*}\Omega/\mathcal{L}^{\perp})) \xrightarrow{\delta_p} C^{\infty}(A, \Lambda^{p+1}(\mathbb{C} \otimes T^{*}\Omega/\mathcal{L}^{\perp})), \tag{3.5}$$

where $0 \leq p \leq n-1$ and $C^{\infty}(A, \Lambda^{p}(\mathbb{C} \otimes T^{*}\Omega/\mathcal{L}^{\perp}))$ denotes the space of germs of sections of $\Lambda^{p}(\mathbb{C} \otimes T^{*}\Omega/\mathcal{L}^{\perp}))$ at the point A. The cohomolgy groups of (3.5) are denoted by

$$\mathcal{H}^{q}(A, \mathcal{L}\,), \qquad q = 0, \ldots, n. \tag{3.6}$$

Results of a general nature are known only under the assumption that $\mathcal{L}$ is *locally integrable*, i.e., when $\mathcal{L}^{\perp}$ is locally generated by the differentials of $m = N - n$ functions of class C^{∞}. Little is known so far about the groups (3.6); there are complete descriptions in the following cases: i) when $\mathcal{L}$ defines an elliptic structure ([T3]), ii) when the structure $\mathcal{L}$ is real analytic and the e Levi form is non degenerate ([T5]), iii) when $n = 1$ ([NT1]), iv) when the structure $\mathcal{L}$ is real analytic, $m = 1$ and $q = 1$ ([T6]) or $q = n$ ([CH2]). Recently Treves extended the results in [CH2] to the C^{∞} case in [T7] and this technique, related to the study of solutions with compact support [CH1], may shed some light on how to obtain the cohomology groups for other structures, like tubes ([T4],[T8]). Analogously, the results in [T6] where extended to the C^{∞} case [MT]. The problem of finding solutions to (3.2) with compact support is also of interest ([CH1], [HT]) and related to the extension property of Hartogs for solutions of the homogeneous equation.

In the locally integrable case of co-rank one, i.e., when $m = 1$ and there is smooth function defined in a neighborhood of A

$$Z : U_A \longrightarrow \mathbb{C}$$

whose differential generates $\mathcal{L}^{\perp}/U_A$, there is a conjecture of Treves that can be stated as follows:

Conjecture *Given* $1 \leq p \leq n$, $\operatorname{Ker} \delta_p = \operatorname{Im} \delta_{p-1}$ *in the complex (3.5) if the following holds:*

> *For every point B in a neighborhood of A there is a decreasing basis of open neighborhoods of B in U_A, $\{V_\nu\}$ such that, for every $\nu \in Z_+$ and every $\zeta \in \mathbb{C}$ the natural map*
>
> $$H_{p-1}(Z^{-1}\{\zeta\} \cap V_{\nu+1}) \longrightarrow H_{p-1}(Z^{-1}\{\zeta\} \cap V_\nu)$$
>
> *is trivial.*

(here H_* denotes reduced homology with complex coefficients).

The conjecture is valid when $p = 1$ or $p = n$ ([CH2], [T6], [T7], [MT]). The cases $1 < p < n$ remain open.

4. Operators with nonsmooth coefficients

If $P(x, D)$ is a principal type differential operator of order m with smooth coefficients satisfying condition $(\mathcal{P})$, the Beals-Fefferman theorem allows to locally solve the equation $P(x, D)u = f$, $f \in L^2$, with $u \in H^{m-1}$. This is the optimal regularity one can expect of u if $P(x, D)$ is not hypoelliptic. A natural question is: how much regularity must one demand on the coefficients of $P(x, D)$ to obtain the same result, if they are not smooth? Counting on the fingers the number of derivatives used in the known constructive proofs of solvability gives a large number that grows with the dimension. This has the following explanation. The microlocalization technique involves the continuity in Sobolev spaces of pseudo-differential and Fourier integral operators and it is a fact of life that the number of derivatives needed to control their norms grows linearly with the dimension n of the surrounding space.

Consider, however, a differential operator of order one with smooth complex coefficients

$$\frac{\partial}{\partial t} + \sum_{j=1}^{n} a_j(x, t) \frac{\partial}{\partial x_j} + c(x, t) \tag{4.1}$$

defined in a neighborhood of the origin of $\mathcal{R}^{n+1}$. After a local change of variables, it can be put in the form

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^{n} b_j(x, t) \frac{\partial}{\partial x_j} + c(x, t) \tag{4.2}$$

with $b_j(x, t)$ smooth and real.

The solvability in L^2 (notice that $m = 1$ in this example) for the operator (4.2) verifying $(\mathcal{P})$ holds if the coefficients in the principal part are Lipschitz and $c(x, t)$ is measurable [Hol], independently of n. Since the rectification of the bicharacteristics needed to put (4.1) into the form (4.2) takes up one derivative, the former is locally solvable in L^2 if the coefficients of the principal part have Lipschitz first-order derivatives. It is not known whether it is possible (or impossible) to bound the regularity of the coefficients for higher order operators with a constant independent of n in order to obtain solvability. A related question is to determine the minimal number of derivatives of a positive symbol (say classical of order one) needed to guarantee that the corresponding pseudo-differential operator verifies a Gårding inequality.

5. Solvability in L^p

Consider again a principal type differential operator $P(x,D)$ of order m with smooth coefficients satisfying condition $(\mathcal{P})$. For a given $1 < p < \infty$ we say that $P(x,D)$ is locally solvable in L^p if we may locally solve the equation $P(x,D)u = f$, $f \in L^p$, with $u \in H_p^{m-1} = (I-\Delta)^{-(m-1)/2}L^p$. This is true if $p = 2$ by the Beals-Fefferman theorem but false, in general, for $p \neq 2$ [Gu] even if $P(x,D)$ is subelliptic. Other examples are due to E. Perdigão (unpublished work in progress) who also announced that there is solvability in L^p for operators in two variables (any order) or of order one (any number of variables) satisfying $(\mathcal{P})$.

The main "explanation" for this fenomena is technical. The reduction technique of Nirenberg-Treves (present in some form in all proofs of solvability) involves a Fourier integral operator which is bounded in L^2 but not necessarily bounded in L^p for $p \neq 2$. When the order is one or the number of variables is two the use of Fourier integral operators can be avoided.

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