Revista de la
Unión Matemática Argentina
Volumen 37, 1991.

H-SYSTEMS WITH VARIABLE H

E. LAMI DOZO and M. C. MARIANI

Abstract. Given a continuous function $H : \mathbf{R}^3 \to \mathbf{R}$, we consider the H-System

$$\text{(H)} \qquad \Delta X = 2H(X)X_u \wedge X_v \quad \text{in } B$$

for an unknown $X : B = \{(u,v) \in \mathbf{R}^2;\ u^2 + v^2 < 1\} \to \mathbf{R}^3$ and we are interested in finding solutions and their properties under boundary conditions.

The main motivation is that for a parametrised disc-type surface in $\mathbf{R}^3$, given an image $X(B)$, supposing moreover, that the coordinates (u, v) are isothermal, i.e.

$$\text{(Iso)} \quad |X_u| - |X_v| = X_u \cdot X_v = 0 \quad \text{in } B$$

then X satisfies (H). This equation is also called the *prescribed mean curvature equation* [10] and the references.

The main problem is, for given H, to find a disc-type surface wich is supported by a given curve in $\mathbf{R}^3$: *Plateau's problem*

For a better understanding of this problem we have studied (H) under Dirichlet or Neumann boundary condition and also boundary properties of solutions in the Sobolev space $H^2(B;\mathbf{R}^3)$. [7],[8] and[9].

NOTATIONS. As in [1], we denote $W^{k,p}(B;\mathbf{R}^3)$ the usual Sobolev spaces and $H^k(B;\mathbf{R}^3) = W^{k,2}(B;\mathbf{R}^3)$.

For $X \in H^1(B;\mathbf{R}^3)$, $\|X\|_{L^2(\partial B;\mathbf{R}^3)} = \left(\int_{\partial B} |TrX|^2\right)^{1/2}$ where Tr is the usual trace operator from $H^1(B;\mathbf{R}^3)$ into $L^2(\partial B;\mathbf{R}^3)$. We recall that if $X \in H^2(B;\mathbf{R}^3)$ then $TrX \in H^1(\partial B;\mathbf{R}^3)$.

For any bounded or essentially bounded function $Y : U \in \mathbf{R}^m \to \mathbf{R}^n$ we will denote $\|Y\|_\infty = \operatorname{ess\,sup}_{w \in U} |Y(w)|$, for example

$$\|H\|_\infty = \sup_{\xi \in \mathbf{R}^3} |H(\xi)|, \quad \|H(X)\|_\infty = \operatorname{ess\,sup}_{w \in B} |H(X(w))|$$

Finally (r, σ) denote the polar coordinates in $\mathbf{R}^2$.

1. The Dirichlet problem

Given $g \in H^1(B;\mathbf{R}^3)$, we still denote g its trace on ∂B and look for an element $X \in H^1(B;\mathbf{R}^3)$ such that

$$\text{(Dir)} \begin{cases} \Delta X = 2H(X)X_u \wedge X_v & \text{in } B \\ X = g & \text{on } \partial B \end{cases}$$

i.e. $X - g \in H_0^1(B;\mathbf{R}^3)$ and that satisfies (H) in the following weak sense

$$\text{(Sol)} \qquad \int_B (\nabla X \cdot \nabla \varphi + 2H(X)X_u \wedge X_v \cdot \varphi) = 0$$

for any $\varphi \in C_0^1(B;\mathbf{R}^3)$.

Denote $Q \in C^1(\mathbf{R}^3;\mathbf{R}^3)$ the vector field defined by

$$Q(\xi) = \left(\int_0^{\xi_1} H(s,\xi_2,\xi_3)\, ds,\ \int_0^{\xi_2} H(\xi_1,s,\xi_3)\, ds,\ \int_0^{\xi_3} H(\xi_1,\xi_2,s)\, ds \right)$$

which satisfies $\operatorname{div} Q = 3H$.

The following functional

$$D_H(X) = D(X) + 2V(X)$$

with $D(X) = \frac{1}{2}\int_B |\nabla X|^2$ the *Dirichlet integral* and

$$V(X) = \frac{1}{3}\int_B Q(X) \cdot X_u \wedge X_v$$

the *Hildebrandt volume* [6], will give weak solutions as critical points of D_H on special convex subsets of $H^1(B;\mathbf{R}^3)$.

For $H \equiv H_0$ a constant in $\mathbf{R}$ and $g \equiv g_0$ also a constant, then $X \equiv g_0$ is the only solution of the Dirichlet problem [13]. If g is non constant with $0 \leq |H_0| \|g\|_\infty < 1$, there are two weak solutions, a *local minimum* of D_H in $T \equiv g + H_0^1(B;\mathbf{R}^3)$, called a *stable solution* and another weak solution which is not a local mimimum in T called an *unstable solution* [3], [11].

For H variable, H in $C^1(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$, and H near $H_0 \in \mathbf{R} \setminus \{0\}$ for the following distance

$$[H - H_0] = \sup(1 + |\xi|)(|H(\xi) - H_0| + |\nabla H(\xi)|) + |Q(\xi) - H_0\xi| + |dQ(\xi) - H_0\, id|$$

defined in [12], it is asserted that given a local minimum of D_{H_0} in T there are two critical points of D_H in T for H in a dense subset of a neighborhood of H_0 [12] or in a full neighborhood [4].

We first find a global minimum of D_H in T for H far from a constant.

THEOREM 1.1 [7]. *Let $H : \mathbf{R}^3 \to \mathbf{R}$ be continuous and bounded. If the function $Q \in C^1(\mathbf{R}^3, \mathbf{R}^3)$ associated to H satisfies that $\|Q\|_\infty < 3/2$ and $\dfrac{\partial Q_i}{\partial \xi_j} \in L^\infty(\mathbf{R}^3)$ for $i \neq j$, $i, j = 1, 2, 3$, then, given $g \in H^1(B; \mathbf{R}^3)$ the functional D_H has a minimum $\underline{X}$ in T and $\underline{X}$ is a weak solution of (Dir).*

A function $g \in H^1(B; \mathbf{R}^3)$, harmonic in B, may be a local minimum of D_H or not. The following result analysis the second possibility.

THEOREM 1.2 [7]. *Let $g \in C^1(\overline{B}, \mathbf{R}^3)$ be harmonic in B. Suppose that $H \in C^1(\mathbf{R}^3) \cap W^{1,\infty}(\mathbf{R}^3)$ and g satisfy*

i) $0 < \|H\|_\infty \|g\|_\infty < 3/2$ and $Q \in L^\infty(\mathbf{R}^3, \mathbf{R}^3)$

ii) For some $c > \|\nabla g\|_\infty$, for all $\xi \in \mathbf{R}^3$.

$$|H(\xi)| \leq \lambda_1^2 / c^2 (|\xi| - \|g\|_\infty)_+$$

where $\lambda_1 > 0$ is the first eingenvalue of $-\Delta$ in $H_0^1(B)$.

Then g is a weak solution of (Dir) and either g is a local minimum of D_H in $T \cap W^{1,\infty}(B; \mathbf{R}^3)$ or there exists a sequence (X_n) in $W^{1,\infty}(B; \mathbf{R}^3)$ of weak solutions of (Dir) with $X_n \to g$ in $W^{1,\infty}(B; \mathbf{R}^3)$.

To search other solutions we use a variant of the Mountain Pass Lemma [10][2].

Consider for $k > 0 \in \mathbf{R}$

$$M(k) = \{X \in T;\ \|\nabla(X - g\|_\infty < k\}$$

and the *slope ρ of D_H in $\overline{M(k)}$, defined by*

$$\rho(X) = \sup\{dD_H(X)(X - Y);\ Y \in \overline{M(k)}\}$$

where

$$dD_H(X)(\varphi) = \int_B (\nabla X \cdot \nabla\varphi + 2H(X)X_u \wedge X_v \cdot \varphi)$$

(Lemma 1 in [7]*).*

THEOREM 1.3 [7]. *Let $H \in L^\infty(\mathbf{R}^3) \cap C(\mathbf{R}^3)$, $Q \in L^\infty(\mathbf{R}^3, \mathbf{R}^3)$ and suppose that $X_0 \in W^{1,\infty}$ is a local minimum of D_H in T. If for some $k > 0$ in $\mathbf{R}$ there exists $X_1 \in \overline{M(k)}$ such that $D_H(X_1) < D_H(X_0)$, then*

$$\beta = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} D_H(\gamma(t))$$

where $\Gamma = \{\gamma : [0,1] \to \overline{M(k)},\ \gamma$ continuous, $\gamma(0) = X_0,\ \gamma(1) = X_1\}$, is attained by D_H in $\overline{M(k)}$ and for some $X \in \overline{M(k)}$ with $D_H(X) = \beta$, we have $\rho(X) = 0$.

This X with 0 slope might be a solution. Under some additional assumption on X this is the case.

THEOREM 1.4 [7]. *If X is in $M(k)$ and $\rho(X) = 0$ then X is a weak solution of the Dirichlet problem.*

Concerning the condition $|H_0|\|X\|_\infty \leq 1$ for the case H constant [10] we have a result which recalls it, and also insures that slope 0 gives solutions.

THEOREM 1.5 [7]. *Let $g \in C^1(\overline{B}, \mathbf{R}^3)$ be harmonic in B. If M is a nonempty convex subset of T such that*

i) For some $\delta > 0$, $g + \varphi \in M$ for any $\varphi \in C_0^1(B; \mathbf{R}^3)$ with $\|\nabla\varphi\|_\infty \leq \delta$.

ii) $c = \sup_{X\in M} \|\nabla X\|_\infty < +\infty$.

And if $H \in C^1(\mathbf{R}^3) \cap W^{1,\infty}(\mathbf{R}^3)$ satisfies for all $\xi \in \mathbf{R}^3$,

$$c^2|H(\xi)| \leq \lambda_1^2(|\xi| - \|g\|_\infty)$$

where $\lambda_1 > 0$ is the first eingenvalue of $-\Delta$ in $H_0^1(B)$, then any $X \in M$ with slope $\rho(X) = 0$ is a weak solution of (Dir).

2. The Neumann problem

We consider as data, $f \in C^1(\partial B; \mathbf{R}^3)$ and we search solutions of

$$(N)\begin{cases} \Delta X = 2H(X)X_u \wedge X_v & \text{in } B \\ \dfrac{\partial X}{\partial n} = f & \text{on } \partial B \end{cases}$$

We call a *weak solution of (N)* a function $X \in H^2(B; \mathbf{R}^3)$ such that

$$(\mathrm{Sol})\begin{cases} \int_B \nabla X \cdot \nabla\varphi + 2H(X)X_u \wedge X_v \cdot \varphi & \text{for all } \varphi \in C_0^1(B; \mathbf{R}^3) \\ Tr\left(\dfrac{\partial X}{\partial r}\right) = f \end{cases}$$

A necessary condition is given by

Theorem 2.1 [8]. *Let $H : \mathbf{R}^3 \to \mathbf{R}$ continuous and bounded, and suppose that $X \in H^2(B, \mathbf{R}^3)$ is a weak solution of (N) verifying $\|H(X)X\|_\infty < 1$.*

Then,

$$D(X) \leq \frac{\|X\|_{L^2(\partial B, \mathbf{R}^3)} \|f\|_{L^2(\partial B, \mathbf{R}^3)}}{2(1 - \|H(X)X\|_\infty)}$$

We only obtain solutions near an harmonic solution g of (N) when this g is not a local minimum of D_H, as in Theorem 1.2.

Theorem 2.2 [8]. *Let $f \in C^1(\partial B, \mathbf{R}^3)$ and suppose that $H : \mathbf{R}^3 \to \mathbf{R}$ is a function satisfying the following properties:*

i) $H \in C^1(\mathbf{R}^3) \cap W^{1,\infty}(\mathbf{R}^3)$ *and* $Q \in L^\infty(\mathbf{R}^3, \mathbf{R}^3)$

ii) There exists $g \in W^{2,\infty}(B; \mathbf{R}^3)$, *$g$ harmonic in B, verifying that* $0 < \|H\|_\infty \|g\|_\infty < 3/2$ *and* $\dfrac{\partial g}{\partial n} = f$, *and a positive number* $c > \|\nabla g\|_\infty$ *such that*

$$|H(\xi)| \leq \lambda_1^2/c^2(|\xi| - \|g\|_\infty)_+ \quad \textit{for all } \xi \in \mathbf{R}^3$$

where $\lambda_1 > 0$ is the first eingenvalue of $-\Delta$ in $H_0^1(B)$.

Then g is a weak solution of (N) and either g is a local minimum of D_H in $W^{1,\infty}(B; \mathbf{R}^3) \cap \{X \in H^2(B; \mathbf{R}^3);\ Tr\ \dfrac{\partial X}{\partial r} = f,\ Tr\ X = Tr\ g\}$, or *there exists a sequence* (X_n) *in* $W^{1,\infty}(B;\mathbf{R}^3)$ *of weak solutions of* (N) *with* $X_n \to g$ *in* $W^{2,\infty}(B;\mathbf{R}^3)$.

Similarly to the uniqueness type result of **[13]** for the Dirichlet problem we have

THEOREM 2.3 [8]. *Let* $X \in H^2(B,\mathbf{R}^3)$ *be a weak solution of* (N) *and suppose that* $H \in \mathbf{R}$ *and* $f = 0$. *Then,* X *is a constant.*

3. Solutions in $H^2(B;\mathbf{R}^3)$

The results for the Neumann problem induces to suspect that we should search for natural boundary conditions to the prescribed mean curvature equation

$$\text{(H)} \qquad \Delta X = 2H(H)X_u \wedge X_v$$

LEMMA 3.1 [9]. *Let* $X \in H^2(B;\mathbf{R}^3)$ *and* $\varphi \in C^1(\overline{B},\mathbf{R}^3)$, *and suppose that* $Q \in L^\infty(\mathbf{R}^3,\mathbf{R}^3)$ *and* $\dfrac{\partial Q_i}{\partial \xi_j} \in L^\infty(\mathbf{R}^3)$ *for* $i \neq j$, $i,j = 1,2,3$. *Then, the directional derivative* $dD_H(X)(\varphi)$ *at* X *in the direction* φ *is given by*

$$dD_H(X)(\varphi) = \int_B [\nabla X \cdot \nabla \varphi + 2H(X)X_u \wedge X_v \cdot \varphi] - 2/3 \int_{\partial B} Q(X) \wedge \frac{\partial X}{\partial \sigma} \cdot \varphi d\sigma$$

Concerning the boundary behaviour of this critical points of D_H *in* $H^2(B;\mathbf{R}^3)$ *we have*

THEOREM 3.1 [9]. *Suppose that* $\dfrac{\partial Q_i}{\partial \xi_j} \in L^\infty(\mathbf{R}^3)$ *for* $i \neq j$ $i,j = 1,2,3$ *and* $Q \in L^\infty(\mathbf{R}^3,\mathbf{R}^3)$, *and let* $X \in H^2(B;\mathbf{R}^3)$ *such that* $dD_H(X)(\varphi) = 0$ $(\varphi \in C^1(\overline{B},\mathbf{R}))$. *Then,* X *is a weak solution of* (H) *and* $\dfrac{\partial X}{\partial \eta} = 2/3Q(X) \wedge \partial X/\partial\sigma$ *in* $L^2(\partial B,\mathbf{R}^3)$, *and we deduce that* $\dfrac{\partial X}{\partial \eta} \cdot \dfrac{\partial X}{\partial \sigma} = 0$ a.e. *in* ∂B.

Surpringly, the additional condition for Plateau's problem (ISO) is automatically satisfied by these solutions.

THEOREM 3.2 [9]. *Suppose that* $Q \in L^\infty(\mathbf{R}^3, \mathbf{R}^3)$, $\dfrac{\partial Q_i}{\partial \xi_j} \in L^\infty(\mathbf{R}^3)$ *for* $i \neq j$, $i, j = 1,2,3$, *and let* $X \in H^2(B; \mathbf{R}^3)$ *such that* $dD_H(X)(\varphi) = 0$ $(\varphi \in C^1(\overline{B}, \mathbf{R}^3))$.

Then, the coordinates (u, v) *are isothermal, i.e.:*

$$|X_u| - |X_v| = X_u \cdot X_v = 0 \qquad \text{a.e. in } B$$

There is another result of the boundary behaviour.

THEOREM 3.3 [9]. *Let* $X \in H^2(B; \mathbf{R}^3)$ *be a solution of* (H) *in* B. *Then,*

$$\int_B \left[|\nabla X|^2 + \left(\frac{|\nabla X|^2}{2}\right)_r r + 2H(X)X_r \cdot X_u \wedge X_v\right] = \int_{\partial B} \left|\frac{\partial X}{\partial \eta}\right|^2 d\sigma \geq 0$$

4. Related problems

In each case we have obtained sufficient conditions to have solutions. They show that necessary conditions when $H \equiv H_0 \in \mathbf{R}$ as $|H_0|\|g\|_\infty \leq 1$ for Plateau's problem [10],[5], are not necessarily fulfilled when H varies. We would like to have some obstruction on classes of H to obtain solutions.

All result on Plateau's problem for variable H would be of interest to us.

References

[1] Adams, Robert: Sobolev Spaces, Academic Press.

[2] Ambrosetti, A. - Rabinowitz, P.H.: Dual variational methods in critical point theory and applications, J-Functional Analysis 14 (1973) 349-381.

[3] Brezis, H. - Coron, J.M.: Multiple solutions of H-Systems and Rellich's conjeture, Comm. Pure Appl. Math. 37 (1984), 149-187.

[4] Guofang W.: The Dirichlet Problem for the equation of prescribed mean curvature, Preprint.

[5] Heinz, E.: On the nonexistence of a surface of constant curvature with finite area and prescribed rectifiable boundary, Arch. Rat. Mech. Anal. 35 (1969), 249-252.

[6] Hildebrandt, S.: Randwertprobleme für Flächen mit Vorgeschriebenes mittlerer Krümmung und Anwendungen auf die kapillaritätstheorie Teil I. Fest vorgegebener Rand. Math - Z. 112 (1969), 205-213.

[7] Lami Dozo, E. - Mariani, M.C.: A Dirichlet Problem for an H-System with variable H, Preprint N° 181 IAM.

[8] Lami Dozo, E. - Mariani, M.C.: The prescribed mean curvature equation with Neumann condition, Preprint N° 182 IAM.

[9] Lami Dozo, E. - Mariani, M.C.: Soluciones de la ecuación $\Delta X = 2H(X)X_u \wedge X_v$ en $H^2(B, \mathbf{R}^3)$, Preprint N°. 180 IAM.

[10] Struwe, M.: Plateau's problem and the calculus of variations, Lecture Notes, Princeton University Press.

[11] Struwe, M.: Non-uniquenes in the Plateau problem for surfaces of constant mean curvature, Arch. Rat. Mech. Anal. 93 (1986), 135-157.

[12] Struwe, M.: Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature, Preprint.

[13] Wente, H.C.: The differential equation $\Delta X = 2H X_u \wedge X_v$ with vanishing boundary values, Proc. AMS 50 (1975), 59-77.