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WEIGHTS FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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1. Introduction

The Hardy-Littlewood maximal opertor is defined for functions $f \in L^1_{loc}(\mathbb{R})$ by

$$Mf(x) = \sup_{a<x<b} \frac{1}{b-a} \int_a^b |f|.$$

Analogously, we can consider the left and the right Hardy-Littlewood maximal operators:

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f| \qquad \text{and} \qquad M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f|.$$

Assume $1 \leq p < \infty$. In 1972, B. Muckenhoupt [Mu] characterized the pairs of weights (u, v), $u, v \geq 0$, such that for all $f \in L^p(v)$ and $\lambda > 0$

$$\int_{\{Mf>\lambda\}} u \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p v$$

where C is a positive constant independent of f and λ (letter C will always mean in this paper a positive constant not necessarily the same at each occurrence). He proved that (1) holds if and only if the pair (u, v) satisfies A_p which means that there exists $C > 0$ such that

$$\sup_{a,b} \frac{1}{b-a} \int_a^b u \left(\frac{1}{b-a} \int_a^b v^{\frac{-1}{p-1}} \right)^{p-1} \leq C \qquad \text{if } p > 1$$

and

$$Mu \leq Cv \quad a.e. \qquad \text{if } p = 1.$$

He also studied the problem for $u = v$ and $p > 1$. He showed that

$$\int_{-\infty}^{\infty} |Mf|^p v \leq C \int_{-\infty}^{\infty} |f|^p v$$

with C independent of f is equivalent to the fact that v, i.e. the pair (v, v), satisfies A_p.

This left open the corresponding problems for M^- and M^+ which are of interest in Ergodic Theory (see for instance [MT1]). E. Sawyer [S2] obtained the following results for M^+ (and similar for M^-).

Theorem A. *Let $1 \le p < \infty$. The following are equivalent:*

(a) There exists $C > 0$ such that for all $f \in L^p(v)$ and $\lambda > 0$

$$\int_{\{M^+f>\lambda\}} u \le \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p v.$$

(b) The pair u, v satisfies A_p^+, i.e., there exists $C > 0$ such that for all $a < b < c$

$$\frac{1}{c-a}\int_a^b v \left(\frac{1}{c-a}\int_b^c v^{\frac{-1}{p-1}}\right)^{p-1} < C \quad \text{if} \quad p > 1,$$

$$M^- v(x) \le K v(x) \quad \text{a.e.} \quad \text{if} \quad p = 1.$$

Theorem B. *If $1 < p < \infty$, v,i.e. the pair (v, v) satisfies A_p^+ if and only if there exists $C > 0$ such that for all $f \in L^p(v)$*

$$\int_{-\infty}^{\infty} |M^+ f|^p v \le C \int_{-\infty}^{\infty} |f|^p v.$$

The proofs of the necessity of the conditions A_p^+ in both theorems are easy and similar to the corresponding ones for A_p classes. However, the proofs of the other implications of the theorems do not follow the ideas of [Mu] and [CF].

In order to prove Theorem A, Sawyer uses the corresponding result for the Hardy operator ([AM],[S3])

$$Tf(x) = \frac{1}{x}\int_0^x f.$$

We may observe that the proofs for T are not very easy. On the other hand, the proof of Theorem B depends on the characterization of the pairs of weights (u, v) for which the inequality

$$\int_{-\infty}^{\infty} |M^+ f|^p u \le C \int_{-\infty}^{\infty} |f|^p v$$

holds. (The result for M was proved by Sawyer [S1]). Both facts are not very nice because

1) the proof of the weighted weak type inequality (Theorem A) is considerably more difficult than the corresponding one for M, and

2) the proof of the strong type inequality (Theorem B) does not seem good to continue the study of M^+ in other spaces like $L^{p,q}$ or Orlicz spaces.

Other proofs of these theorems for more general one-sided operators have been given in [MOT] and [A] but the difficulties pointed out in 1) and 2) remain.

2. New Proofs

Recently, new proofs of Theorems A and B have been given in [M]. These proofs follow the pattern of Muckenhoupt's case ([Mu], [CF]). First it is proved Theorem A in a simple way and then Theorem B follows from the fact that $A_p^+ \Rightarrow A_{p-\varepsilon}^+$ which is the most difficult step. The first aims of this paper are:

1) To give a very simple proof of Theorem A, different of the one in [M].

2) To present a scheme of the proof of $A_p^+ \Rightarrow A_{p-\varepsilon}^+$.

In order to prove that A_p^+ implies the weighted weak type inequality (p, p) we consider a function f, and numbers x and $h > 0$ fixed. Then we choose a decreasing sequence $\{x_k\}$

such that

$$x_0 = x + h \quad \text{and} \quad u(x_{k+1}, x_k) = u(x, x_{k+1}),$$

where, as usual, $u(a,b) = \int_a^b u$. Then, by Hölder inequality and the fact that (u, v) satisfies A_p^+,

$$\begin{aligned}\int_x^{x+h} |f| &= \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} |f| \le \sum_{k=0}^{\infty} \left(\int_{x_{k+1}}^{x_k} |f|^p v^p \right)^{\frac{1}{p}} \left(\int_{x_{k+1}}^{x_k} v^{-\frac{1}{p-1}} \right)^{\frac{p-1}{p}} \\ &\le C \sum_{k=0}^{\infty} \frac{\left(\int_{x_{k+1}}^{x_k} |f|^p v \right)^{\frac{1}{p}}}{(u(x_{k+2}, x_{k+1}))^{\frac{1}{p}}} (x_k - x_{k+2}) \\ &\le Ch (M_u^+(|f|^p v u^{-1})(x))^{\frac{1}{p}},\end{aligned}$$

where

$$M_u^+ f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f| u}{\int_x^{x+h} u}.$$

Therefore, we have

$$M^+ f(x) \le C(M_u^+(|f|^p v u^{-1})(x))^{\frac{1}{p}}.$$

This inequality and the fact that M_u^+ is of weak type $(1,1)$ with respect to the measure $u(x)dx$ give the desired weighted weak type inequality in Theorem A.

As we announced above, in order to prove Theorem B, the key fact is the implication $A_p^+ \Rightarrow A_{p-\varepsilon}^+$. In this moment, it is convenient to recall how this fact is normally proved in the case of A_p classes. First, it is seen that $v \in A_p$ implies that v satisfies the following Reverse Hölder Inequality:

$$\left(\frac{1}{b-a} \int_a^b v^{1+\delta} \right)^{\frac{1}{1+\delta}} \le C \frac{1}{b-a} \int_a^b v$$

for some positive constants C and δ independent of the numbers a and b. Then, as a corollary, $v \in A_p$ and the Reverse Hölder Inequality give easily that $v \in A_{p-\varepsilon}$. But now , the Reverse Hólder Inequality does not hold for A_p^+ classes (consider, for instance, $v(x) = \exp x$). However, a substitute has been found in [M]: if $v \in A_p^+$ then there exist positive constants C and δ such that for all a and b

$$\int_a^b v^{1+\delta} \le C(M(v\chi_{(a,b)})(b))^{\delta} \int_a^b v.$$

which implies

$$Mv^{1+\delta} \le C(M(v\chi_{(a,b)})(b))^{1+\delta}.$$

This is what we have called Weak Reverse Hölder Inequality. This condition together with $v \in A_p^+$ give $v \in A_{p-\varepsilon}$ in [M] but not so easily as in the classical case of Muckenhoupt's classes. In the proof of these inequalities, we use, as in Theorem A, the method of cutting intervals with respect to some function but the proofs are considerably harder than the one we proved above. As a resume, we could say that what this process of cutting intervals does is to put the problem in a suitable way for applying the technics we know about the

Muckenhoupt A_p classes.

Until now, we have presented here new proofs of known results. However, perhaps, the important thing is that these new proofs have allowed to obtain new results for M^+ and to know better A_p^+ classes. In this way it has been posible to study M^+ on weighted $L_{p,q}$ and Orlicz spaces (see [O1], [O2], [OP]). In what follows, we will show results in other direction.

3. A_∞^+ WEIGHTS

After proving $A_p^+ \Rightarrow A_{p-\varepsilon}$ the following questions remained open: It is known that Reverse Hölder Inequality is equivalent to the fact that the weight is in some A_p class. Is this true for the Weak Reverse Hölder Inequality and A_p^+ classes? Moreover, is there a concept of A_∞^+ weights, equivalent to the Weak Reverse Hölder Inequality, analogous to the concept of A_∞ weights?

The answers to these questions is affirmative. This has been obtained recently by L. Pick, A. de la Torre and the author ([MPT])(see also [GP]).

To introduce the concept of A_∞^+ weights, assume that $v \in A_p^+$ and let $a < b < c$ and $E \subset (b,c)$. By Hölder's Inequality and A_p^+

$$|E|^p \leq \int_E v \left(\int_b^c v^{-\frac{1}{p-1}}\right)^{p-1} \leq C\frac{v(E)}{v(a,b)}(c-a)^p$$

where $v(E) = \int_E v$. Therefore we have

$$\frac{|E|}{c-a} \leq C\left(\frac{v(E)}{v(a,b)}\right)^{\frac{1}{p}}.$$

Keeping in mind this inequality, we define A_∞^+ weights.

DEFINITION. *If $v > 0$ is a locally integrable function, we say that $v \in A_\infty^+$ or v satisfies A_∞^+ if there exist positive constants C and δ such that for all $a < b < c$ and all $E \subset (b,c)$ we have*

$$\frac{|E|}{c-a} \leq C\left(\frac{v(E)}{v(a,b)}\right)^{\delta}.$$

Some of the main results about A_∞^+ [MPT] are collected in the following theorem.

THEOREM C. *The following are equivalent:*

(a) $v \in A_\infty^+$.

(b) There exist positive constants C and δ such that for all $a < b < c$ and all $E \subset (a,b)$ we have

$$\frac{v(E)}{v(a,c)} \leq C\left(\frac{|E|}{c-b}\right)^{\delta}.$$

(c) There exist $p > 1$ such that $v \in A_p^+$.

(d) There exist positive constants C and δ such that for all $a < b < c$

$$\int_a^b v^{1+\delta} \leq C(M(v\chi_{(a,b)})(b))^\delta \int_a^b v.$$

The equivalence between (a), (c) and (d) answer to the questions we made above, while the equivalence between *(a)* and *(b)* is nothing but the one-sided version of the comparability of measures introduced in [CF]. In fact, the equivalence of *(a)* and *(b)* allows to use A_∞^+ to obtain weighted distribution function inequalities. For example, consider the one-sided fractional operator (Weyl operator) and the one-sided fractional maximal operators.Let $0<\alpha<1$. We define

$$I_\alpha^+ f(x)=\int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}dy \qquad \text{and} \qquad M_\alpha^+ f(x)=\sup_{h>0}\frac{1}{h^{1-\alpha}}\int_x^{x+h}|f(y)|dy.$$

It is clear that $M_\alpha^+ f\le I_\alpha^+|f|$. For $v\in A_\infty^+$ one proves

$$\int_{\{M_\alpha^+ f>2\lambda, I_\alpha^+ f\le\gamma\lambda\}} v\le C\gamma\int_{\{I_\alpha^+ f>\lambda\}} v,$$

for some C and γ, and then, as it is well known, this good λ-inequality gives

$$\int_{-\infty}^\infty |I_\alpha^+ f|^p v\le C\int_{-\infty}^\infty |M_\alpha^+ f|^p v.$$

The details can be found in [MPT].

4. One-Sided BMO Spaces

In what follows we will introduce a one-sided sharp maximal function that will play the role of the classical sharp maximal function.

It is well known that for a real locally integrable function f in the real line, the sharp maximal function $f^\sharp$ is defined at x by

$$f^\sharp(x)=\sup_{x\in I}\frac{1}{|I|}\int_I\left|f(y)-\frac{1}{|I|}\int_I f\right|dy$$

where the supremum is taken over all the bounded intervals containing x. If f is such that $f^\sharp\in L^\infty$ we say that f is a function of bounded mean oscillation and we write

$$BMO=\{f\in L^1_{loc}: f^\sharp\in L^\infty\}.$$

There is a close relation between BMO and A_p weights. More precisely, for fixed $p>1$,

$$BMO=\{\alpha\log v: v\in A_p, \alpha\ge 0\}.$$

Next we introduce one-sided sharp functions and one-sided BMO spaces.

Definition. *If f is a locally integrable function in the real line, we define*

$$f_+^\sharp(x)=\sup_{h>0}\frac{1}{h}\int_x^{x+h}\left(f(y)-\frac{1}{h}\int_{x+h}^{x+2h} f\right)^+ dy$$

where $z^+=\max(z,0)$.

We say that a function f is in BMO^+ if $f_+^\sharp\in L^\infty$ and we write

$$\|f\|_{*,+}=\|f_+^\sharp\|_\infty.$$

Observe that BMO^+ is not a vector space, increasing functions are obviously in BMO^+ and $\|f\|_{*,+}=0$ if and only if $f(x)\le f(y)$ for almost all x and y with $x\le y$. It is also clear that $f,g\in BMO^+$ and $\alpha\ge 0$ imply $f+g\in BMO^+$ and $\alpha f\in BMO^+$. Furthermore, it is not difficult to see that if $v\in A_p^+$ and $\alpha\ge 0$ then $\log v^\alpha$ belongs to BMO^+. Then a first question arises: are all the functions in BMO^+ the logarithm of v^α for some $\alpha>0$

and some $v \in A_\infty^+$? The answer is affirmative ([MT3]), but before of establishing the corresponding theorem we will go through another question.

From now on, if I is an interval (a,c) we write $I^- = (a,b)$ and $I^+ = (b,c)$ where $b = (a+c)/2$.

It is easy to see that

$$f_+^\sharp(x) \le \sup_{h>0} \inf_a \frac{1}{h}\int_x^{x+h} (f(y)-a)^+ dy + \frac{1}{h}\int_{x+h}^{x+2h} (a-f(y))^+ dy.$$

Therefore, if for each interval I there exists a number a_I such that

$$\sup_I \frac{1}{|I^-|}\int_{I^-} (f-a_I)^+ + \frac{1}{|I^+|}\int_{I^+} (a_I - f)^+ < \infty$$

then $f \in BMO^+$. Then it is natural to ask if the converse is also true. The answer is affirmative ([MT3]) but it is not clear for the author how to prove this using only the definitions. In fact, it was needed a kind of John-Niremberg Inequality in [MT3] to prove it. This inequality is in the following lemma. Before stating the lemma, we introduce some notation. If $I = (a,d)$ is an interval and $f$is a locally integrable function then we write $f_I = \frac{1}{|I|}\int_I f$, $I^l = (a,b)$, $I^c = (b,c)$ and $I^r = (c,d)$ where $b-a = c-b = d-c$.

LEMMA. *There exist constants $C > 0$ and $\alpha > 0$ such that for every $f \in BMO^+$, every interval I and all $\lambda > 0$*

$$|\{x \in I^l : (f(x) - f_{I^r})^+ > \lambda\}| \le C|I| \exp\left(\frac{-\alpha\lambda}{\|f\|_{*,+}}\right).$$

A similar result in the context of parabolic partial differential equations appears in [FG]. Using this lemma we have the following theorem which answers to the above questions.

THEOREM D. *Let $1 < p < \infty$. The following statements are equivalent:*

(a) $f \in BMO^+$.

(b) There exists $\gamma > 0$ such that $\exp(\gamma f) \in A_\infty^+$.

(c) There exists $\gamma > 0$ such that $\exp(\gamma f) \in A_p^+$.

(d) For every interval I there exists a_I such that

$$\sup_I \frac{1}{|I^-|}\int_{I^-} (f-a_I)^+ + \frac{1}{|I^+|}\int_{I^+} (a_I - f)^+ < \infty.$$

Once we have this theorem, it is possible to improve the lemma in the sense that one can avoid the lack between I^l and I^r. Then we have another version of one-sided John-Niremberg Inequality.

THEOREM E (JOHN NIREMBERG INEQUALITY). *Let $f \in BMO^+$. Then there exist positive constants C and α such that for every interval I and all $\lambda > 0$*

$$|\{x \in I^- : (f(x) - f_{I^+})^+ > \lambda\}| \le C|I| \exp\left(\frac{-\alpha\lambda}{\|f\|_{*,+}}\right).$$

Let us come back to the classical case. It is clear that $f^\sharp \le 2Mf$. There is not an oppossite pointwise inequality but there is an oppossite weighted integral inequality which is important in the study of integral sigular operators, (for instance,see [GR]). The same things happens with M^+ and $f_+^\sharp$. On one hand, $f_+^\sharp \leq 3M^+f$. On the other hand we have the following theorem.

THEOREM F. *Assume* $v \in A_\infty^+$, $f \geq 0$ *and* $M^+f \in L^{p_0}(v)$ *for some* p_0, $0 < p_0 < \infty$. *Then for every* p, $p_0 \leq p < \infty$,

$$\int_{-\infty}^{\infty} (M^+f)^p v \leq K \int_{-\infty}^{\infty} {f_+^\sharp}^p v.$$

This theorem can be applied to the study of the weights for the one-sided fractional integrals. Observe that $M_\alpha^+ f \leq I_\alpha^+|f|$. This inequality can be reversed using the one-sided sharp maximal funcion. More precisely, we have

$$(I_\alpha^+ f)_+^\sharp \leq C_\alpha M_\alpha^+ f.$$

This inequality together with Theorem F give roughly speaking that $\int_{-\infty}^{\infty} |I_\alpha^+ f|^p v$ and $\int_{-\infty}^{\infty} |M_\alpha^+ f|^p v$ are comparable. Then the study of the weights for I_α^+ is reduced to the case of M_α^+. In this way it is possible to characterize the good weioghts for I_α^+ and M_α^+ (see the details in [MT3]). We may point out here that these charaxterizations were proved by the first time in [AS] (see also [MT2]).

5. Open Questions

1) It is known that the good weights for the Hilbert transform are the same as the good ones for the Hardy-Littlewood maximal operator (see [HMW], [CF], [GR]). Does there exist a kind of integral singular which plays the same role with respect to M^+?

2) All what we have written here is refered to the real line. It is natural to try to generalize this study to $\mathbb{R}^n$, $n > 1$. For instance, which are the good weights for the maximal operator defined in $\mathbb{R}^2$ by

$$M^{++}f(x,y) = \sup_{h>0} \frac{1}{h^2} \int_x^{x+h} \int_y^{y+h} |f|$$

for locally integrable functions in $\mathbb{R}^2$?

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