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Infinite Dimensional Homogeneous Reductive Spaces

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Abstract

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{\times}$ the group of invertible elements of $\mathcal{A}$. A general setting for the theory of homogeneous reductive spaces $\mathcal{Q}$ over $\mathcal{A}^{\times}$ is presented. Also, in the case where $\mathcal{A}$ is a C*-algebra, we present the notion of an involution in $\mathcal{Q}$. The differential geometry associated to such spaces is developed.

1 Introduction.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{\times}$ the group of invertible elements of $\mathcal{A}$. Several examples of homogeneous reductive spaces (see[9,10] for the definitions in the finite dimensional case) with group $\mathcal{A}^{\times}$, are studied in[1,3,4,5,6,13]. In the case where $\mathcal{A}$ is a C*-algebra, these spaces present a natural involution and a Finsler structure. Metric properties of these spaces are studied in the mentioned articles and are based on results of[2,8,16].

We present here some basic results, of a theory, where the aforementioned examples find a natural treatment. We also present the concept of the *classifying connection* on homogeneous reductive spaces and show some of its properties (section 6).

We study the general concept of *involution* in a homogeneous reductive space in the case when $\mathcal{A}$ is a C*-algebra and, in connection with it, we define the notion of *regularity*. The main result in relation with the idea of regularity is Theorem 10.1. The hypothesis of regularity has been verified in all finite dimensional cases and some infinite dimensional ones (see[7]).

Some of these results were presented in the "Tercera Escuela Venezolana de Matemáticas", Mérida 1990, see[12]. The general setting for this theory, appeared for the first time in the preprint[13] which was submitted for publication in *Acta Científica Venezolana*. The last part of this preprint studies the Space of representations of a compact Lie Group on a C*-algebra as an example for this theory. Here we ommit most of the proofs (they appear in complete detail in[13]) still, we include the proof of Theorem 10.1.

2 Notation, Definitions and Constructions.

Let $\mathcal{Q}$ be a Banach manifold, and $\mathcal{L}$ a smooth locally transitive left action of $\mathcal{A}^\times$ on $\mathcal{Q}$. For $\varepsilon \in \mathcal{Q}$, let π_ε be the smooth mapping $\pi_\varepsilon : \mathcal{A}^\times \to \mathcal{Q}$ given by the action on $\varepsilon \in \mathcal{Q}$, i.e. $\pi_\varepsilon(g) = \mathcal{L}_g(\varepsilon),\ \forall g \in \mathcal{A}^\times$.

Let $I\!I_\varepsilon = \pi_\varepsilon^{-1}(\varepsilon) \subset \mathcal{A}^\times$ be the isotropy subgroup of ε by the action of $\mathcal{A}^\times$; $I\!I_\varepsilon$ is a submanifold of $\mathcal{A}^\times$. The mapping $\pi_\varepsilon : \mathcal{A}^\times \to \mathcal{Q}$ is a principal fiber bundle with total space $\mathcal{A}^\times$ and group $I\!I_\varepsilon$. The action of $I\!I_\varepsilon$ is right-multiplication.

We identify $\mathcal{A}$ with the tangent space of $\mathcal{A}^\times$ at the identity, $T_1(\mathcal{A}^\times)$, and $\mathcal{I}_\varepsilon = T_1(I\!I_\varepsilon)$ shall denote the Lie algebra of the group $I\!I_\varepsilon$.

Let us denote with $\tilde{\pi}_\varepsilon = T_1(\pi_\varepsilon) : \mathcal{A} = T_1(\mathcal{A}^\times) \to T_\varepsilon(\mathcal{Q})$. For any $g \in \mathcal{A}^\times$, let $\phi_g = \mathcal{R}_{g^{-1}} \circ \mathcal{L}_g : \mathcal{A}^\times \to \mathcal{A}^\times$, i.e. $\phi_g(h) = ghg^{-1}, \forall h \in \mathcal{A}^\times$. The tangent mapping $(T\phi_g)_1 : \mathcal{A} \to \mathcal{A}$ shall be denoted with $\tilde{\phi}_g$ (also called $Ad(g)$ in[10]).

We present connections on the principal fiber bundles $\pi_\varepsilon : \mathcal{A}^\times \to \mathcal{Q},\ \forall \varepsilon \in \mathcal{Q}$, so the morphisms

$$\begin{array}{ccc} \mathcal{A}^\times & \xrightarrow{\phi_g} & \mathcal{A}^\times \\ \pi_\varepsilon \downarrow & & \downarrow \pi_{\mathcal{L}_g(\varepsilon)} \\ \mathcal{Q} & \xrightarrow{\mathcal{L}_g} & \mathcal{Q} \end{array}$$

are isomorphisms of principal fiber bundles with connections.

Consider any mapping $K_\varepsilon : T_\varepsilon(\mathcal{Q}) \to \mathcal{A} (= T_1(\mathcal{A}^\times))$, with the following two properties:

(I) $\tilde{\pi}_\varepsilon \circ K_\varepsilon : T_\varepsilon(\mathcal{Q}) \to T_\varepsilon(\mathcal{Q})$ is the identity mapping.

(II) $(T\phi_a)_1 : \mathcal{A} \to \mathcal{A}$, leaves invariant the subspace $\mathcal{H}^\varepsilon = K_\varepsilon(T_\varepsilon(\mathcal{Q})) \subset \mathcal{A},\ \forall a \in I\!I_\varepsilon$, i.e. $(T\phi_a)_1(\mathcal{H}^\varepsilon) = \mathcal{H}^\varepsilon,\ \forall a \in I\!I_\varepsilon$.

The existence K_ε is equivalent to having a connection on the principal bundle $\pi_\varepsilon : \mathcal{A}^\times \to \mathcal{Q}$ which makes $\mathcal{Q}$ into a *homogeneous reductive space* as defined in[11]. To produce such a connection for $\pi_\varepsilon : \mathcal{A}^\times \to \mathcal{Q}$, from K_ε, we construct a smooth distribution of horizontal spaces $\mathcal{H}$ as follows: for each $\varepsilon \in \mathcal{Q}$ and $g \in \mathcal{A}^\times$ define $\mathcal{H}_g^\varepsilon = g\mathcal{H}^\varepsilon = \{a \in \mathcal{A} \mid a = gh, \text{ for some } h \in \mathcal{H}^\varepsilon\}$. We call the elements of $\mathcal{H}_g^\varepsilon$ the *horizontal vectors* at $g \in \mathcal{A}^\times$ (for π_ε). Notice that for $1 \in \mathcal{A}^\times$, $\mathcal{H}_1^\varepsilon = \mathcal{H}^\varepsilon = K_\varepsilon(T_\varepsilon(\mathcal{Q}))$. The distribution $\mathcal{H}$ on $\mathcal{A}^\times$ is given by these horizontal spaces, $\mathcal{H}_u = \mathcal{H}_u^\varepsilon = u\mathcal{H}^\varepsilon$ for $u \in \mathcal{A}^\times$. The following assertion is immediate:

Claim 2.1 *The spaces $\mathcal{H}_u$, constitute a connection on the principal bundle $\pi_\varepsilon : \mathcal{A}^\times \to \mathcal{Q}$, i.e.*

(i) *$\mathcal{H}$ is a smooth distribution on $\mathcal{A}^\times$.*

(ii) $u\mathcal{A} = (T(\mathcal{A}^\times))_u = \mathcal{V}_u \oplus \mathcal{H}_u, \forall\, u \in \mathcal{A}^\times$, *with* $\mathcal{V}_u = u\mathcal{I}_\varepsilon$.

(iii) $\mathcal{H}_{ua} = \mathcal{H}_u a\, (= (\mathcal{R}_a)_* \mathcal{H}_u), \forall\, a \in I\!I_\varepsilon, u \in \mathcal{A}^\times$.

Properties (i), (ii) and (iii) define a connection on $\pi_\varepsilon : \mathcal{A}^\times \rightarrow \mathcal{Q}$ as found in[10] p. 63.

The connections in the principal bundles π_ε's induce connections on several associated bundles, in particular on the tangent bundle $T\mathcal{Q} \rightarrow \mathcal{Q}$, which we present in section 4. First we shall construct the $\mathcal{A}$-valued 1-form $\mathcal{K}$ on $\mathcal{A}^\times$ induced by K_ε

2.1 Construction of the 1-form $\mathcal{K}$.

Definition: The 1-form $\mathcal{K}$ is defined by

$$\mathcal{K}(\mu) = \tilde{\phi}_g \circ K_\varepsilon \circ (\tilde{\mathcal{L}}_g)^{-1}, \text{ if } \mathcal{L}_g(\varepsilon) = \mu$$

This 1-form has the equivariance property described below. We remark the following two lemmas. Let $\mu = \mathcal{L}_g(\varepsilon) \in \mathcal{Q}$.

Lemma 2.2 $\pi_\mu \circ \phi_g(h) = \mathcal{L}_g \circ \pi_\varepsilon(h), \forall\, h \in \mathcal{A}$.

Lemma 2.3 (Infinitesimal version of Lemma 2.2)

$$\tilde{\pi}_\mu \circ \tilde{\phi}_g = \tilde{\mathcal{L}}_g \circ \tilde{\pi}_\varepsilon, \text{ and} \tag{1}$$

$$\tilde{\phi}_g(\mathcal{I}_\varepsilon) = \mathcal{I}_\mu. \tag{2}$$

Given an $\mathcal{A}$-valued 1-form $\mathcal{K}$, we say that $\mathcal{K}$ *splits* the action of $\mathcal{A}^\times$ on $\mathcal{Q}$ if $K_\mu = \mathcal{K}(\mu) : T_\mu(\mathcal{Q}) \rightarrow \mathcal{A}$, satisfies the two conditions (I) and (II) for $K_\mu\ \forall\, \mu \in \mathcal{Q}$, as above for K_ε. Suppose $\mathcal{K}$ splits the $\mathcal{A}^\times$ action, we say that $\mathcal{K}$ is *equivariant* if it satisfies:

$$K_\mu \circ \tilde{\mathcal{L}}_g = \tilde{\phi}_g \circ K_\varepsilon \tag{3}$$

$$\forall\, g \in \mathcal{A}^\times \text{ and } \forall\, \varepsilon, \mu \in \mathcal{Q}, \text{ with } \mu = \mathcal{L}_g(\varepsilon)$$

Lemma 2.4 *If $\mathcal{K}$ is equivariant, then $\tilde{\phi}_g(\mathcal{H}^\varepsilon) = \mathcal{H}^\mu$ where $\mathcal{H}^\varepsilon = K_\varepsilon(T_\varepsilon(\mathcal{Q}))$, $\forall\, \varepsilon \in \mathcal{Q}$, and $\mu = \mathcal{L}_g(\varepsilon)$.*

Observation: an equivariant 1-form as above is completely determined (locally) by $K_\varepsilon = \mathcal{K}(\varepsilon)$ at any fixed $\varepsilon \in \mathcal{Q}$. This follows from the hypothesis of (local) transitivity for the action $\mathcal{L}$, i.e. $\forall\, \varepsilon, \mu \in \mathcal{Q}$, (close enough) $\exists g \in \mathcal{A}^\times$ such that $\mu = \mathcal{L}_g(\varepsilon)$, and then the equivariance of $\mathcal{K}$ implies that

$$K_\mu = \tilde{\phi}_g \circ K_\varepsilon \circ (\tilde{\mathcal{L}}_g)^{-1} \tag{4}$$

Lemma 2.5 *K_μ is well defined, i.e. it is independent of g with $\mu = \mathcal{L}_g(\varepsilon)$.*

By the equation 4, we may consider the principal fiber bundle associated to μ, $\pi_\mu : \mathcal{A}^\times \to \mathcal{Q}$, via the left action of $\mathcal{A}^\times$ on $\mathcal{Q}$, i.e. $\pi_\mu(g) = \mathcal{L}_g(\mu)$, $\forall g \in \mathcal{A}^\times$.

Lemma 2.6 *The mapping $K_\mu : T_\varepsilon(\varepsilon) \to \mathcal{A}$, satisfies (I) and (II) as above, i.e.*

(I) $\tilde{\pi}_\mu \circ K_\mu : T_\mu(\mathcal{Q}) \to T_\mu(\mathcal{Q})$ *is the identity mapping.*

(II) $(T\phi_a)_1 : \mathcal{A} \to \mathcal{A}$, *leaves invariant the subspace* $\mathcal{H}^\mu = K_\mu(T_\mu(\mathcal{Q})) \subset \mathcal{A}$ $\forall\, a \in I\!I_\mu$, *i.e.* $(T\phi_a)_1(\mathcal{H}^\mu) = \mathcal{H}^\mu, \forall\, a \in I\!I_\mu$.

3 Parallel Transport and The Transport Equation.

Let K_ε and $\mathcal{K}$ equivariant as in the previous section. To construct the connection ∇_K on the tangent bundle of $\mathcal{Q}$ we introduce the *transport equation* whose solutions give the horizontal lifts to $\mathcal{A}^\times$ of curves on $\mathcal{Q}$ for π_ε.

Given a smooth curve $\gamma(t)$ on $\mathcal{Q}$, with $\gamma(0) = \varepsilon$ and $t \in I$ (I = an interval about zero).

Definition: The differential equation

$$\dot{\Gamma}(t) = K_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t) \tag{5}$$

is called the transport equation for $\gamma(t)$, and the solution $\Gamma(t)$ with the initial condition $\Gamma(0) = 1 \in \mathcal{A}^\times$, is called the *horizontal lift* of $\gamma(t)$ (for π_ε).

Proposition 3.1 $\forall\, t \in I$

$$\pi_\varepsilon(\Gamma(t)) = \gamma(t) \tag{6}$$
$$\dot{\Gamma}(t) \in \mathcal{H}^\varepsilon_{\Gamma(t)} \tag{7}$$

4 The Covariant Derivative ∇_K.

Let $\gamma(t)$ be a smooth curve on $\mathcal{Q}$ with $\gamma(0) = \varepsilon$ and t in some interval about zero. Let $\Gamma(t)$ be the solution of the differential (transport) equation (5) $\dot{\Gamma}(t) = K_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t)$, with initial condition $\Gamma(0) = 1$. Let $Y(t)$ be a tangent (smooth) field along $\gamma(t)$ on $\mathcal{Q}$. To define the covariant derivative $\frac{D^K Y}{dt}(0) \in T_\varepsilon(\mathcal{Q})$, we just define $K_\varepsilon\left(\frac{D^K Y}{dt}(0)\right) \in \mathcal{A}$ and the covariant derivative given by $\frac{D^K Y}{dt}(0) = \tilde{\pi}_\varepsilon \circ K_\varepsilon\left(\frac{D^K Y}{dt}(0)\right) \in \tilde{M}_\varepsilon$. We use the following notation: for any tangent vector $X \in \tilde{\mathcal{Q}}_\varepsilon$, we write $\hat{X} = K_\varepsilon(X) \in \mathcal{H}_\varepsilon$.

Definition:

$$\frac{D^K Y}{dt} = \tilde{\pi}_\varepsilon\left(\widehat{\frac{D^K Y}{dt}}\right), \text{ where } \widehat{\frac{D^K Y}{dt}}(0) = \frac{d}{dt}\left\{K_\varepsilon\left[(T\mathcal{L}_{\Gamma(t)})^{-1}Y(t)\right]\right\}_{|t=0}.$$

Proposition 4.1 *The covariant derivative* D^K *induces the connection* ∇_K *on* TQ *given by the formula*

$$\widehat{\nabla_{KX}Y} = \dot{\hat{Y}} + [\hat{Y}, \hat{X}] \tag{8}$$

where $\dot{\hat{Y}} = X(\hat{Y})$.

5 The Classifying Connection ∇_C.

We introduce the connection ∇_C on TQ which has the same geodesics of ∇_K *but with opposite torsion*, so $\nabla_P = \frac{1}{2}(\nabla_K + \nabla_C)$ satisfies:

Theorem 5.1
(i) ∇_P *is symmetric.*
(ii) ∇_P *has the same geodesics as* ∇_K *(and* ∇_C*)*

Remark: In the finite dimensional case, ∇_P is the only connection on TQ satisfying (i) and (ii), according to[11].

The proof of Theorem 5.1 follows from the results presented below.

5.1 The Space P of Projections in $\mathcal{A}$.

Let $\mathbf{P} = \mathbf{P}(\mathcal{A})$ denote the submanifold of $End(\mathcal{A}) = \{$endomorphisms of $\mathcal{A}\}$, given by

$$\mathbf{P} = \left\{q \in End(\mathcal{A}) | q^2 = q\right\} = \{ \text{ projections of } \mathcal{A} \ \}$$

and $\pi : \xi \to \mathbf{P}$, the canonical fiber bundle $\xi \subset \mathbf{P} \times \mathcal{A}$ with fiber $\pi^{-1}(q) = \{(q, a) \in \mathbf{P} \times \mathcal{A} \mid q(a) = a\}$. The canonical connection[15] is given as follows: let $\eta(t)$ be a curve in $\mathbf{P}$ and σ a section (along η) of the bundle $\pi : \xi \to \mathbf{P}$, i.e. $\sigma(t) = (\eta(t), a(t))$ with $a(t) \in \mathrm{Img}(\eta(t)) \subset \mathcal{A}$. We define the covariant derivative

$$\frac{D\sigma}{dt}(0) = (\eta(0), \eta(0)(a')), \text{ where } a' = \frac{d}{dt}(a(t))_{|_{t=0}} \in \mathcal{A}.$$

5.2 The Classifying Map Π.

The family of projections $\Pi_\varepsilon = K_\varepsilon \circ \tilde{\pi}_\varepsilon : \mathcal{A} \to \mathcal{A}$ for $\varepsilon \in Q$, defines a differentiable mapping (observe $\Pi_\varepsilon^2 = \Pi_\varepsilon$) $\Pi : Q \to \mathbf{P} \subset End(\mathcal{A})$ with, $\Pi(\varepsilon) = \Pi_\varepsilon : \mathcal{A} \to \mathcal{A}$. We call Π the classifying map[15]. We present a formula for the tangent mapping of Π at $\varepsilon \in Q$. For $x \in \mathcal{A}$ let's denote by $\mathrm{ad}_x : \mathcal{A} \to \mathcal{A}$, given by

$$\mathrm{ad}_x(a) = [x, a] = xa - ax, \ \ \forall a \in \mathcal{A}.$$

Proposition 5.2 *For* $\varepsilon \in Q$ *and* $X \in (TQ)_\varepsilon$, $(T\Pi)_\varepsilon(X) = \left[\mathrm{ad}_{K_\varepsilon(X)}, \Pi_\varepsilon\right]$ *as operators on* $\mathcal{A}$.

5.3 The Induced Connection ∇_C on TQ.

We use the classifying map Π to construct the covariant derivative[15] on TQ. First, consider the following commutative diagram,

$$\begin{array}{ccc} TQ & \overset{(P,K)}{\underset{\tilde{\pi}_\varepsilon \circ P_2}{\Longleftrightarrow}} & \xi \subset \mathbf{P} \times \mathcal{A} \\ \downarrow & & \downarrow \pi \\ Q & \overset{\Pi}{\longrightarrow} & \mathbf{P} \end{array}$$

where the mapping P is defined by following arrows from TQ down to $\mathbf{P}$, and P_2 gives the second component of an element in ξ. Let $\gamma_t = \gamma(t)$ be a smooth curve on Q with $\gamma(0) = \varepsilon$, and let $Y_t = Y(t)$ be a smooth vector field along γ. To define $\frac{\mathrm{D^C}Y}{dt}$ consider $\eta(t) = \pi(\gamma_t)$ in $\mathbf{P}$ and the section $\sigma(t)$ of the bundle $\pi : \xi \to \mathbf{P}$ given by

$$\sigma(t) = (\eta(t), K_{\gamma_t}(Y_t)) \in \xi_{\eta(t)}$$

We define

$$\frac{\mathrm{D^C}Y}{dt} = \tilde{\pi}_\varepsilon \circ P_2(\frac{D\sigma}{dt})$$

with $\frac{D\sigma}{dt}$ defined above in section 5.1.

5.4 A Formula for ∇_C.

We have $\frac{\mathrm{D^C}Y}{dt} = \tilde{\pi}_\varepsilon(\Pi_\varepsilon(a'))$ where $a' = \frac{d}{dt}(K_{\gamma_t}(Y_t))_{|t=0}$, $(a' \in \mathcal{A})$ but $\tilde{\pi}_\varepsilon \circ \Pi_\varepsilon = \tilde{\pi}_\varepsilon$, so we get $\frac{\mathrm{D^C}Y}{dt} = \tilde{\pi}_\varepsilon((K_{\gamma_t}(Y_t))_{|t=0}) = \tilde{\pi}_\varepsilon(\dot{\hat{Y}})$. Hence we can compute $\widehat{\frac{\mathrm{D^C}Y}{dt}} = K_\varepsilon(\frac{\mathrm{D^C}Y}{dt})$, to get,

$$\widehat{\frac{\mathrm{D^C}Y}{dt}} = \Pi_\varepsilon(\dot{\hat{Y}}) \tag{9}$$

In general if X and Y are vector fields on Q, we have the following formula for the connection ∇_C at $\varepsilon \in Q$.

$$\widehat{\nabla_{C}}_X Y = \Pi_\varepsilon(\dot{\hat{Y}}) \tag{10}$$

6 The Geodesics of ∇_K and ∇_C.

The following lemmas are useful in a proof of Theorem 5.1.

Lemma 6.1 *The geodesics of ∇_K and ∇_C coincide.*

Proof: It is enough to check that the geodesics are given by the same differential equation in both connections. In fact, by Proposition 4.1, a geodesic γ of ∇_K is given by the equation,

$$0 = \frac{\mathrm{D^K}\dot{\gamma}}{dt} = \tilde{\pi}_\varepsilon(\dot{\hat{\gamma}}) + [\hat{\gamma}, \dot{\hat{\gamma}}] = \tilde{\pi}_\varepsilon(\dot{\hat{\gamma}})$$

On the other hand, γ is a geodesic of ∇_C when (see formula (9)),

$$0 = \frac{\mathrm{D^C}\dot{\gamma}}{dt} = \tilde{\pi}_\varepsilon(\dot{\hat{\gamma}})$$

and Lemma 6.1 is proved.

Observation: If ∇_1, ∇_2 are connections and s, t real numbers with $s+t = 1$ then $s\nabla_1 + t\nabla_2$ is a connection. Furthermore if γ is a geodesic for both ∇_1 and ∇_2, then γ is a geodesic of $s\nabla_1 + t\nabla_2$.

6.1 Computation of the Torsions of ∇_K and ∇_C.

Let X and Y be tangent vectors to Q, say at ε. We consider particular vector fields on Q extending X and Y. Given $x \in \mathcal{A}$, consider the smooth vector field on Q given by

$$\underline{x}(\mu) = \tilde{\pi}_\mu(x), \quad \forall\, \mu \in Q \tag{11}$$

Observe that $K_\mu(\underline{x}(\mu)) = \Pi_\mu(x)$, $\forall\, \mu \in Q$. If $x = K_\varepsilon(X)$ and $y = K_\varepsilon(Y)$, the vector fields $\underline{x}$ and $\underline{y}$ extend X and Y respectively.

Claim 6.2 *If $x = K_\varepsilon(X)$ and $y = K_\varepsilon(Y)$, $\tilde{\pi}_\varepsilon([x, y]) = -[\underline{x}, \underline{y}](\varepsilon)$.*

To compute $\mathrm{T^K}(X, Y)(\varepsilon)$, the torsion of ∇_K, it is enough to compute $\widehat{\mathrm{T^K}}(X, Y)(\varepsilon) = K_\varepsilon(\mathrm{T^K}(X, Y)(\varepsilon))$. From the formula for the torsion,
$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, we have the following identity,

$$\widehat{\mathrm{T^K}}(X, Y) = K_\varepsilon(\nabla_{K\underline{x}}\,\underline{y}(\varepsilon)) - K_\varepsilon(\nabla_{K\underline{y}}\,\underline{x}(\varepsilon)) - K_\varepsilon([\underline{x}, \underline{y}](\varepsilon))$$

but from claim 6.2 we have $[\underline{x}, \underline{y}](\varepsilon) = -\tilde{\pi}_\varepsilon[x, y]$ and we get $\widehat{\mathrm{T^K}}(X, Y) = K_\varepsilon(\nabla_{K\underline{x}}\,\underline{y}(\varepsilon)) - K_\varepsilon(\nabla_{K\underline{y}}\,\underline{x}(\varepsilon)) + \Pi_\varepsilon([x, y])$. We shall compute $\widehat{\nabla_{K\underline{x}}}\,\underline{y}(\varepsilon) = K_\varepsilon(\nabla_{K\underline{x}}\,\underline{y}(\varepsilon)) = \underline{x}(\hat{\underline{y}})(\varepsilon) + [x, y]$ (the last equality follows from the formula in Proposition 4.1).

Claim 6.3 $\underline{x}(\underline{y})(\varepsilon) = (I - \Pi_\varepsilon)[x, y]$.

Now we can write $\widehat{\nabla_{K\underline{x}}}\,\underline{y}(\varepsilon) = \underline{x}(\hat{\underline{y}})(\varepsilon) + [x, y] = (I - \Pi_\varepsilon)[x, y] - [x, y] = -\Pi_\varepsilon([x, y])$ and similarly $\widehat{\nabla_{K\underline{y}}}\,\underline{x}(\varepsilon) = -\Pi_\varepsilon([y, x])$, hence we get $\widehat{\mathrm{T^K}}(X, Y) = -\Pi_\varepsilon([x, y]) - \Pi_\varepsilon([x, y]) + \Pi_\varepsilon([x, y])$, hence

$$\widehat{\mathrm{T^K}}(X, Y) = -\Pi_\varepsilon([x, y]) \tag{12}$$

Finally, from formula (10) and claim 6.3 we have,
$\widehat{\nabla_C}(\underline{x},\underline{y})(\varepsilon) = \Pi_\varepsilon(\underline{x}(\hat{y})(\varepsilon)) = \Pi_\varepsilon(I - \Pi_\varepsilon)([x,y]) = 0$ and, the torsion T^C of ∇_C is given by, $\widehat{T^C}(X,Y) = -K_\varepsilon([\underline{x},\underline{y}]) = \Pi_\varepsilon([x,y])$, where the last equality follows from claim 6.2, hence we get the formula,

$$\widehat{T^C}(X,Y) = \Pi_\varepsilon([x,y]) \tag{13}$$

6.2 The Proof of Theorem 5.1.

Part (ii) follows from Lemma 6.1 and the observation following that lemma. To prove part (i) observe that the torsion T^P of ∇_P satisfies, $T^P = \frac{1}{2}(T^K + T^C)$. But by the formulas (12) and (13) we get $T^P = 0$, hence ∇_P is symmetric as claimed.

7 Formulas for the Geodesics and the Exponential Mapping.

Lemma 7.1 *The geodesic $\gamma(t)$ through $\varepsilon \in Q$ at $t = 0$ and $\dot{\gamma}(0) = X \subset (TQ)_\varepsilon$ is given by*

$$\gamma(t) = \mathcal{L}_{\exp(tK_\varepsilon(X))}\varepsilon \tag{14}$$

Hence the exponential mapping at $\varepsilon \in Q$ is given by

$$\exp_\varepsilon(X) = \mathcal{L}_{\exp(K_\varepsilon(X))}\varepsilon \tag{15}$$

8 The Curvatures R^K and R^C.

For tangent vectors X, Y, Z at $\varepsilon \in Q$, the curvature of a connection ∇ on TM is given by the formula (as in p. 133 of[14])

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

We shall present the formulas after acting with K_ε on both sides of this formula for the curvature to get,

$$\hat{R}(X,Y)Z = \hat{\nabla}_X\nabla_Y Z - \hat{\nabla}_Y\nabla_X Z - \hat{\nabla}_{[X,Y]}Z$$

8.1 The Formula of R^K.

$$\widehat{R^K}(X,Y)Z = \Big[K(Z), (I - \Pi)[K(X), K(Y)]\Big] \tag{16}$$

8.2 The Formula of R^C.

$$\widehat{R^C}(X,Y)Z =$$
$$= \Big[K(X),(I-\Pi)[K(Y),K(Z)]\Big] - \Big[K(Y),(I-\Pi)[K(X),K(Z)]\Big] \quad (17)$$

When working on the formula above, it is useful and interesting to notice the following assertion, which follows from the hypothesis that $\tilde{\phi}_a$ leaves $\mathcal{H}^\epsilon$ invariant for any $a \in \mathit{\Pi}_\varepsilon$.

Claim 8.1 *If* x $\in \mathcal{H}^\epsilon$ *and* y $\in \mathcal{I}^\epsilon$, *then* $[x,y] \in \mathcal{H}^\epsilon$, *i.e. the bracket of a horizontal* x *with a vertical* y, *is horizontal.*

9 Involutions.

In this section we shall consider $\mathcal{A}$ to be an associative C*-algebra (with identity) and $\mathcal{A}^\times$ its multiplicative group. Let's denote with

$$\Sigma(g) = g^\Sigma = (g^*)^{-1} \quad (18)$$

the action of the *contragradient* involution $\Sigma : \mathcal{A}^\times \to \mathcal{A}^\times$. Observe that $(g^\Sigma)^\Sigma = g$

9.1 Homogeneous Reductive Spaces with Involution.

We shall say that $\sigma : \mathcal{Q} \to \mathcal{Q}$ is an *involution* for the homogeneous reductive space $\{\mathcal{Q},\mathcal{K}\}$ if σ is a diffeomorphism of period two (σ^2 = identity), which satisfies the following four axioms:

Axiom 1. Equivariance of σ.

For any $\varepsilon \in \mathcal{Q}$ the following diagram commutes,

$$\begin{array}{ccc} \mathcal{A}^\times & \xrightarrow{\Sigma} & \mathcal{A}^\times \\ \pi_\varepsilon \downarrow & & \downarrow \pi_{\varepsilon^\sigma} \\ \mathcal{Q} & \xrightarrow{\sigma} & \mathcal{Q} \end{array}$$

i.e. $(\mathcal{L}_g\varepsilon)^\sigma = \mathcal{L}_{g^\Sigma}(\varepsilon^\sigma)$.

Axiom 2. Compatibility with $\mathcal{K}$.

For any $\varepsilon \in \mathcal{Q}$ the following diagram commutes

$$\begin{array}{ccc} (T\mathcal{A}^\times)_1 = \mathcal{A} & \xrightarrow{(T\Sigma)_1} & \mathcal{A} \\ K_\varepsilon \uparrow & & \uparrow K_{\varepsilon^\sigma} \\ (T\mathcal{Q})_\varepsilon & \xrightarrow{(T\sigma)_\varepsilon} & (T\mathcal{Q})_{\varepsilon^\sigma} \end{array}$$

i.e. $(T\Sigma)_1 K_\varepsilon = K_{\varepsilon^\sigma}(T\sigma)_\varepsilon$

Axiom 3. The set $\mathcal{P}$ of *self-adjoint* elements of $\mathcal{Q}$,

$$\mathcal{P} = \{\varepsilon \in \mathcal{Q} | \varepsilon^{\sigma} = \varepsilon\}$$

is a submanifold of $\mathcal{Q}$ and the (connected) components of $\mathcal{P}$ are the intersection of $\mathcal{P}$ with the components of $\mathcal{Q}$.

Axiom 4. For any $\varepsilon \in \mathcal{Q}$, there exists $a = a(\varepsilon) \in \mathcal{A}^{+}$ such that $\mathcal{L}_a \varepsilon = \varepsilon^{\sigma}$.

Observations:

1. Axiom 1 implies that $\Sigma : \mathcal{A}^{\times} \longrightarrow \mathcal{A}^{\times}$ maps $\mathbb{I}_{\varepsilon} \longrightarrow \mathbb{I}_{\varepsilon^{\sigma}}$

2. From Axiom 1, for $\rho \in \mathcal{P}$, the Lie algebra $\mathcal{I}_{\rho}$ (of $\mathbb{I}_{\rho}$) is a C^*-subalgebra of $\mathcal{A}$.

3. Axiom 2 implies that $(T\Sigma)_1 : \mathcal{A} \longrightarrow \mathcal{A}$, maps $u \mapsto -u^*$, hence $\mathcal{H}^{\varepsilon^{\sigma}} = \{u^* | u \in \mathcal{H}^{\varepsilon}\}$.

4. From Axiom 2, for $\rho \in \mathcal{P}$, the horizontal space $\mathcal{H}^{\rho}$ is $*$-closed in $\mathcal{A}$, and we have the decomposition

$$\mathcal{H}^{\rho} = (\mathcal{H}^{\rho})^{\mathrm{s}} \oplus (\mathcal{H}^{\rho})^{\mathrm{a}}$$

into symmetric and anti-symmetric elements.

5. For the examples of spaces with involutions we shall present below, the mapping $a : \mathcal{Q} \longrightarrow \mathcal{A}^{+}$ in Axiom 4, can actually be chosen to be smooth. In general Lemma 9.1 below says that given $\varepsilon \in \mathcal{Q}$ we may take the mapping a to be smooth in some neighborhood of ε.

6. Suppose that $\mathcal{L}_a \varepsilon = \varepsilon^{\sigma}$ with $a > 0$ and $\mathcal{L}_g \varepsilon = \mu$ for some $g \in \mathcal{A}^{\times}$, then from axiom 1 we have

$$\begin{aligned} \mu^{\sigma} &= \mathcal{L}_{g^{\Sigma}} \varepsilon^{\sigma} = \mathcal{L}_{(g^*)^{-1}} \mathcal{L}_a \mathcal{L}_{g^{-1}} \mu, \text{ hence} \\ \mu^{\sigma} &= \mathcal{L}_{(g^*)^{-1} a g^{-1}} \mu \end{aligned} \tag{19}$$

Lemma 9.1 *Given $\varepsilon \in \mathcal{Q}$, the mapping $a : \mathcal{Q} \longrightarrow \mathcal{A}^{+}$ in Axiom 4 above can be taken locally smooth about ε.*

Proof of the lemma: Let $V \subset (T\mathcal{Q})_{\varepsilon}$ be a neighborhood of $0 \in (T\mathcal{Q})_{\varepsilon}$ such that the exponential $\exp_{\varepsilon} : V \longrightarrow U$ (see formula (15)) is a diffeomorphism onto a neighborhood $U \subset \mathcal{Q}$ of ε. For $\mu \in U$ we call $g(\mu) = \exp(K_{\varepsilon}(X_{\mu}))$ where $X_{\mu} = \exp_{\varepsilon}^{-1}(\mu) \in V$. Then $g : U \longrightarrow \mathcal{A}^{\times}$ is smooth and $\mathcal{L}_{g(\mu)}\varepsilon = \mu$ hence, from formula (19) above, we have $\mathcal{L}_{\tilde{a}(\mu)}\mu = \mu^{\sigma}$ with $\tilde{a} : U \longrightarrow \mathcal{A}^{\times}$ smooth given by $\tilde{a}(\mu) = (g(\mu)^*)^{-1} a (g(\mu))^{-1}$.

9.2 Decomposition of $(T\mathcal{Q})_\rho$.

Claim 9.2 $X \in (T\mathcal{P})_\rho \Longleftrightarrow K_\rho(X)^* = -K_\rho(X)$.

For $\rho \in \mathcal{P}$ we define $\mathcal{N}_\rho = \{X \in (T\mathcal{Q})_\rho | K_\rho(X)^* = K_\rho(X) \in \mathcal{H}^\rho \subset \mathcal{A}\}$. From claim 9.2, and the fact that $K_\rho(T\mathcal{Q})_\rho = \mathcal{H}^\rho = (\mathcal{H}^\rho)^{\mathrm{s}} \oplus (\mathcal{H}^\rho)^{\mathrm{a}}$, (see the observations above) we can write

$$(T\mathcal{Q})_\rho = (T\mathcal{P})_\rho \oplus \mathcal{N}^\rho \tag{20}$$

10 The Normal Bundle on $\mathcal{P}$.

Let $\{\mathcal{Q}, \mathcal{K}, \sigma\}$ be a homogeneous reductive space with involutión and $\mathcal{P} \subset \mathcal{Q}$ the self-adjoint elements of $\mathcal{Q}$. We denote by $\mathcal{N}$ the *normal bundle* on $\mathcal{P}$

$$\mathcal{N} = \bigcup_{\rho \in \mathcal{P}} \mathcal{N}^\rho$$

We define the exponential mapping $\mathcal{E} : \mathcal{N} \to \mathcal{Q}$ given by $\mathcal{E}(\rho, X) = \exp_\rho(X)$, i.e. the restriction of the exponential map $\exp_\rho$ to normal vectors $X \in \mathcal{N}^\rho$, $K_\rho(X) = K_\rho(X)^* \in \mathcal{A}$. We are interested in giving sufficient conditions for the mapping $\mathcal{E}$ to be a diffeomorphism.

10.1 The Hypothesis of Regularity.

For a fixed $\rho \in \mathcal{P}$, consider $\mathcal{I}_\rho^+$ the set of (invertible) positive elements in the Lie algebra $\mathcal{I}_\rho$ of $I\!I_\rho$. Consider also the set E^ρ of the exponentials of the symmetric elements of $\mathcal{H}^\rho$, i.e. $E^\rho = \{\exp(h) | h \in (\mathcal{H}^\rho)^{\mathrm{s}}\}$. Consider the mapping $p_\rho : \mathcal{I}_\rho^+ \times E^\rho \to \mathcal{A}^+$, given by $p_\rho(\iota, e) = p$, where p is the positive part in the polar decomposition $\iota e = pu$, with $p > 0$ and u unitary.

Definition: We say that ρ is regular if the mapping p_ρ is a diffeomorphism.

Observations:

1. Just by taking the inverses (or the $*$'s) the hypothesis of regularity can be rewritten to say that the mapping $\tilde{p}_\rho : \mathcal{I}_\rho^+ \times E^\rho \to \mathcal{A}^+$ is a diffeomorphism where $\tilde{p}_\rho(\iota, e) = \tilde{p} > 0$ from the polar decomposition $e\iota = u\tilde{p}$.

2. If $\rho \in \mathcal{P}$ is regular, then any element in the unitary orbit of ρ, say $\rho' = \mathcal{L}_u \rho$ with u unitary, is also regular for in such case, $p_{\rho'}(\iota', e') = u(p_\rho(u^{-1}\iota' u, u^{-1}e' u))u^{-1}$, using $u^{-1}\mathcal{I}_{\rho'}^+ u = \mathcal{I}_\rho^+$ and $u^{-1}E^{\rho'} u = E^\rho$, etc...

Definition: We say that the unitary orbit $U_\rho = \{\mathcal{L}_u \rho | u = \text{unitary}\}$ is regular if any element in U_ρ is regular (see observation above).

10.2 The Normal Bundle of a Regular Unitary Orbit.

Let $\mathcal{P}_0 \subset \mathcal{P}$ be a regular unitary orbit as above. Let $\mathcal{N}_0$ denote the restriction to $\mathcal{P}_0$ of the normal bundle $\mathcal{N}$. Let $\mathcal{Q}_0 = \{\mathcal{L}_g\rho | \, g \in \mathcal{A}^\times, \ \rho \in \mathcal{P}_0\}$. Then $\{\mathcal{Q}_0, \mathcal{K}, \sigma\}$ is a homogeneous reductive space with involution, with $\mathcal{P}_0$ the set of self-adjoint elements of $\mathcal{Q}_0$ and, $\mathcal{N}_0$ is the normal bundle of $\mathcal{P}_0$.

Theorem 10.1 *If $\{\mathcal{Q}_0, \mathcal{K}, \sigma\}$ is as above for a regular unitary orbit $\mathcal{P}_0$, then $\mathcal{E}_0 : \mathcal{N}_0 \longrightarrow \mathcal{Q}_0$ is a diffeomorphism.*

Proof of the theorem: To simplify the notation along this proof, we shall drop the subindex '0' so, we start assuming that $\{\mathcal{Q}, \mathcal{K}, \sigma\}$ is the homogeneous reductive space for the unitary orbit $\mathcal{P}$ etc...

Surjectivity. Let $\varepsilon \in \mathcal{Q}$. We want to find $(\rho', X') \in \mathcal{N}$ such that $\mathcal{E}(\rho', X') = \varepsilon$. From axiom 4, there exists $a > 0$ in $\mathcal{A}$ and $\rho \in \mathcal{P}$ such that $\mathcal{L}_a\rho = \varepsilon$. By the regularity hypothesis (for ρ) we can write $a^{-1} = p_\rho(\iota, \exp(h))$ (in a unique way) for u unitary, $\iota \in \mathcal{I}_\rho^+$ and $\exp(h) \in E^\rho$ ($h \in (\mathcal{H}^\rho)^s$) with $a^{-1}u = \iota\exp(h)$, hence $a = u\exp(-h)\iota^{-1}$ and $\varepsilon = \mathcal{L}_{u\exp(-h)\iota^{-1}}\rho = \mathcal{L}_{u\exp(-h)}\rho$, for ι is in the isotropy of ρ. Hence we can write $\varepsilon = \mathcal{L}_{u\exp(-h)u^{-1}}\mathcal{L}_u\rho = \mathcal{L}_{\exp(-uhu^{-1})}\rho'$, with $\rho' = \mathcal{L}_u\rho$. Now, observing that $-uhu^{-1} \in u(\mathcal{H}^\rho)^s u^{-1} = (\mathcal{H}^{\rho'})^s$ and using the formula (15) for the exponential we have $\varepsilon = \exp_{\rho'}(X')$ with $X' = \pi_{\rho'}(-uhu^{-1})$. So we have $\varepsilon = \mathcal{E}(\rho', X')$ as desired.

Injectivity. Suppose (ρ_1, X_1), $(\rho_2, X_2) \in \mathcal{N}$ with $\mathcal{E}(\rho_1, X_1) = \mathcal{E}(\rho_2, X_2)$, i.e. $\mathcal{L}_{e_1}\rho_1 = \mathcal{L}_{e_2}\rho_2$ where $e_1 = \exp(K_{\rho_1}(X_1))$ and $e_2 = \exp(K_{\rho_2}(X_2))$. Consider u unitary with $\mathcal{L}_u\rho_1 = \rho_2$. Then we can write $\mathcal{L}_{e_1^{-1}e_2u}\rho_1 = \rho_1$, hence $e_1^{-1}e_2u \in \mathit{II}_{\rho_1}$ (the isotropy of ρ_1). Consider $e_1^{-1}e_2u = \iota v$ the polar decomposition in the Lie algebra $\mathcal{I}_{\rho_1}$ with v unitary and $\iota > 0$. We can write $e_1^{-1}e_2w = \iota$ with $w = uv^{-1}$ satisfying $\mathcal{L}_w\rho_1 = \rho_2$. Then $e_1^{-1}w(w^{-1}e_2w) = \iota$ hence $w^{-1}e_2w = w^{-1}e_1\iota$ is a positive element in $\exp((\mathcal{H}^{\rho_1})^s)$. By the regularity hypothesis (for ρ_1) such element can be written in a unique way as a product like the one on the right hand side, with w^{-1} unitary, e_1 in $\exp((\mathcal{H}^{\rho_1})^s)$ and $\iota > 0$ in $\mathcal{I}_{\rho_1}$, hence $w^{-1}e_2w = e_1$; $w^{-1} = 1$ and $\iota = 1$, then $\rho_2 = \mathcal{L}_w\rho_1 = \mathcal{L}_1\rho_1 = \rho_1$ and $e_2 = e_1$. Taking logarithms we get $K_{\rho_1}(X_1) = K_{\rho_1}(X_2)$ ($\Longleftrightarrow X_1 = X_2$) as we wanted to show.

Smoothness. We present below a explicit formula for the inverse of $\mathcal{E}$. In turn, the formula for $\mathcal{E}^{-1}$ shows that this mapping is smooth. Consider $\varepsilon \in \mathcal{Q}$ and let $a : U \longrightarrow \mathcal{A}^+$ smooth as in lemma 9.1, i.e., U is a neighborhood of ε and $\mathcal{L}_{a(\mu)}\mu = \mu^\sigma$, $\forall \in U$. Consider $0 < b = \sqrt{a} : U \longrightarrow \mathcal{A}^+$ which is also smooth. Observe that $\mathcal{L}_{b(\mu)}\mu = \rho(\mu) \in \mathcal{P}$, $\forall \mu \in U$ as the following computation shows:

$$\begin{aligned}(\mathcal{L}_{b(\mu)}\mu)^\sigma &= \mathcal{L}_{(b(\mu))^\Sigma}\mu^\sigma = \mathcal{L}_{(b(\mu))^{-1}}\mu^\sigma \\ &= \mathcal{L}_{b(\mu)}\mathcal{L}_{(b(\mu))^{-2}}\mu^\sigma = \mathcal{L}_{b(\mu)}\mu\end{aligned}$$

Clearly $\rho : U \to \mathcal{P}$ is smooth. Consider the decomposition $e(\mu)\iota(\mu) = u(\mu)(b(\mu))^{-1}$, produced from the hypothesis of regularity of $\rho(\mu) \in \mathcal{P}$, so that $\iota(\mu) > 0$ is in $\mathcal{I}_{\rho(\mu)}$ and $e(\mu) = \exp(h(\mu))$ with $h(\mu) \in (\mathcal{H}^{\rho(\mu)})^s$ and, $u(\mu)$ is unitary in $\mathcal{A}$. These mappings ι, e and u are smooth on U, which can be seen immediately from the second observation made above after the definition of 'regularity'. Consider $\tilde{e}(\mu) = u^{-1}(\mu)e(\mu)u(\mu) = \exp(\tilde{h}(\mu)) \in \exp((\mathcal{H}^{\tilde{\rho}(\mu)})^s)$ with $\tilde{\rho}(\mu) = \mathcal{L}_{u^{-1}(\mu)}\rho(\mu)$. Then $\mathcal{L}_{\tilde{e}(\mu)}\tilde{\rho}(\mu) = \mu \; \forall \; \mu \in U$ because

$$\mathcal{L}_{\tilde{e}}\tilde{\rho} = \mathcal{L}_{ueu^{-1}}\mathcal{L}_{u^{-1}}\rho = \mathcal{L}_{u^{-1}e}\rho = \mathcal{L}_{b\iota^{-1}}\rho = \mathcal{L}_{b}\rho = \mathcal{L}_{b(\mu)}\rho(\mu) = \mu$$

Hence $\mathcal{E}^{-1}(\mu) = (\tilde{\rho}(\mu), \tilde{\pi}_{\rho(\mu)}(\tilde{h}(\mu))) \in \mathcal{N}_{\rho(\mu)}$, is the desired formula for the inverse of $\mathcal{E}$.

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